

Hamiltonian Paths in Non-Rectangular Grid Graphs

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by

Hazel J.M. Everett

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**Head
Department of Computational Science
University of Saskatchewan
Saskatoon, Saskatchewan
S7N 0W0**

Hamiltonian Paths in Non-Rectangular Grid Graphs

***Candidate:* Hazel J.M. Everett**

***Supervisor:* J. Mark Keil**

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Abstract

The Hamiltonian path problem is to find a path in a graph from a starting vertex to an ending vertex such that every vertex is visited once. If the starting and ending vertices are the same, the problem becomes the Hamiltonian circuit problem. Both problems are NP-complete. However, for some specialized types of graphs, a polynomial time solution can be found. Itai, Papadimitriou and Szwarcfiter have found that for general grid graphs the problem is NP-complete, but for rectangular grid graphs linear time is sufficient.

The thesis includes a discussion of the sufficient conditions for graphs to be Hamiltonian. Polynomial time algorithms for restricted classes of graphs are presented. Particular attention is paid to the problem for grid graphs. The author's research on the Hamiltonian path problem in non-rectangular grid graphs is presented. This includes

algorithms for finding paths in classes of graphs which are less restricted than rectangular grid graphs: evenly decomposable grid graphs of size four, grid graphs of size six with a near perfect or perfect matching, pyramid grid graphs of size six without holes and L-shaped grid graphs without holes.

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CHAPTER 1

Introduction

Computational problems can be categorized based on the length of time it takes to compute their solutions. Two important classes are P and NP-complete. The best known algorithms to solve the problems in the NP-complete class require time exponential to the size of the problem and thus for all practical purposes NP-complete problems are not computable. Problems in P, on the other hand, require polynomial time which is tolerable in most cases.

It is sometimes possible to place restrictions on a general NP-complete problem to obtain a polynomial time algorithm. For example, in some geometric problems in which the input is a general polygon and the problem is NP-complete, the restriction to a polygon without holes has successfully led to polynomial time algorithms [Lubiw].

There are three motivations for looking at such restricted problems. First, since the exact solution to the problem cannot be found, it may be appropriate to obtain an approximation. One method of obtaining an approximation is to restrict the problem to one that can be solved. Secondly, the restricted problem may be an interesting problem in itself. Finally, this process contributes to the establishment of the distinguishing features of the

P and NP classes.

1.1. The Problem

The Hamiltonian path problem is to find a path in a graph from a starting vertex to an ending vertex such that every vertex is visited once. There are two varieties of the problem depending on whether the starting and ending vertices are specified or not. If the starting and ending vertices are the same the problem becomes the Hamiltonian circuit problem. Both the Hamiltonian path problem and the circuit problem are well known to be NP-complete on general graphs. Polynomial time algorithms have been found for the problem when restrictions are placed on the graph. For example, bipartite permutation graphs [Brandstädt et al.] and interval graphs [Keil] permit polynomial time solutions. For other types of restricted graphs, 3-connected cubic graphs for example, the problem remains NP-complete [Garey, Johnson and Tarjan].

A grid graph is a graph in which vertices lie only on integer coordinates and edges connect vertices that are separated by a distance of one. Grid graphs can be a useful representation in many applications. Myers suggests modelling city blocks in which street intersections are vertices and streets are edges [Myers]. Luccio and Mugnai suggest using a grid graph to represent two-dimensional array type memory accessed by a read/write head moving up, down or across [Luccio et al.]. The vertices correspond to the center of

each cell and edges connect adjacent cells. Finding a path in the grid graph corresponds to accessing all the data. Consider the problem of painting a rectilinear polygon with a paintbrush of a given width without ever lifting the paintbrush. Suppose a general rectilinear polygon is partitioned into cells the width of the paintbrush. To solve the painting problem it is necessary to find a Hamiltonian path in the corresponding grid graph.

Itai, Papadimitriou and Szwarcfiter have shown that the Hamiltonian path problem for general grid graphs, with or without specified endpoints, is NP-complete [Itai et al.]. The problem for rectangular grid graphs, however, is in P requiring only linear time [Itai et al.]. This thesis investigates the complexity of the problem for non-rectangular grid graphs. This problem is difficult so the approach proposed is to consider a sequence of simpler problems. These problems are evenly-decomposable grid graphs, L-shaped grid graphs and pyramid grid graphs. These problems have been chosen because their geometric characteristics lead to simpler solutions. They are not necessarily applicable in themselves. This process of solving geometric problems on simple types of polygons and using the results to help solve the more general problem has been used successfully in the past, see [Wood et al.].

1.2. Overview of the Thesis

The thesis starts with an examination of what is known about the complexity of the Hamiltonian path and circuit problems. Chapter Two defines

terms and introduces some restricted classes of graphs for which the problem remains NP-complete. This includes an examination of the Hamiltonian path problem for general grid graphs. Chapter Three takes a look at the sufficient conditions for the existence of paths and circuits. Algorithms based on these conditions are examined. Chapter Four looks at graphs for which necessary and sufficient conditions have been found that lead to polynomial time algorithms. The algorithm for rectangular grid graphs is examined in detail.

Chapter Five outlines the research on grid graphs. An informal examination of the problem for non-rectangular grid graphs shows the reasoning behind the choice of restricted graphs for which algorithms are presented in Chapters Six through Eight. The restrictions are based on the shape of the graph and the size and parity of the rectangles formed in a decomposition of the graph. The restricted graphs for which algorithms are presented are evenly decomposable grid graphs of size four, grid graphs of size six with a near perfect or perfect matching, pyramid grid graphs of size six without holes and L-shaped grid graphs without holes.

Chapter Nine presents a summary of the work and some suggestions for further research.

CHAPTER 2

The Hamiltonian Path Problem is NP-Complete

2.1. Definition of Terms

This section defines the terms used in the thesis. A reader familiar with the usual terminology, as in [Beineke et al.], may bypass this section.

A *graph*, $G = (V, E)$, is a set of vertices, V , and a set of edges, $E = \{\{i, j\} \mid i \text{ and } j \in V\}$. A *directed graph*, or *digraph*, is a graph with oriented edges. A *planar graph* is a graph that can be embedded in the plane without any crossing edges. A graph is *bipartite* if $V = \{A \cup B\}$ and $E = \{\{a, b\} \mid a \in A \text{ and } b \in B\}$. The *complement* of a graph is a graph with the same vertex set but where two vertices are adjacent if they are not adjacent in the original graph. A graph G to the power k , G^k , is a graph in which two vertices are joined if they are separated by at most distance k in G .

A graph, $G' = (V', E')$, is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. G' is an *induced subgraph* of G if $E' = E \cap \{\{v, w\} \mid v, w \in V'\}$. A graph is called *complete* if there is an edge between each pair of vertices. A maximal complete subgraph, called a *clique*, is a complete subgraph that is not contained in any other complete subgraph. K_n is a complete subgraph with n vertices. $K_{i,j}$ is a complete bipartite graph; $|A| = i$, $|B| = j$, ($|x|$ denotes

the cardinality of x) and every vertex in A is adjacent to every vertex in B .

A *path* in a graph is a sequence $v_0, v_1, \dots, v_k$ such that $\{v_{i-1}, v_i\}$ is an edge for all $1 \leq i \leq k$ and all v_i are distinct. If v_0 and v_k are the same vertices then the path is a *circuit*. If the circuit(path) includes every vertex then the circuit(path) is a *Hamiltonian circuit*(path). A graph with a Hamiltonian circuit is called Hamiltonian. P_n is a path linking n vertices.

A graph is *connected* if there is a path between every pair of vertices. A graph is *$k+1$ -connected* if the removal of any k vertices results in a connected graph. A graph is not *$k+1$ -connected* if the removal of k vertices disconnects the graph. A vertex is a *cut-vertex* if its removal disconnects the graph. The neighborhood of a vertex is the subgraph induced by its adjacent vertices. A graph is *locally connected* if each of its vertices has a connected neighborhood. A graph is *Hamiltonian-connected* if there is a Hamiltonian path connecting every pair of vertices.

The *degree* or *valency* of a vertex is the number of edges incident to it. In a directed graph, the *indegree* of a vertex is the number of edges entering the vertex and the *outdegree* is the number of edges leaving the vertex. The valency of a vertex in a directed graph is the sum of the indegree and the outdegree of the vertex. A graph is *m -regular* if each vertex has valency m .

The edges of a planar graph divide the plane into *faces*. The *interior faces* are bounded by edges on all sides and the *exterior face* is unbounded. The *boundary* of a 2-connected graph is the exterior face.

Two graphs are *isomorphic* if there is a one-to-one mapping between their vertex sets which preserves adjacencies. A graph is *automorphic* if it has a non-trivial isomorphism onto itself. Two graphs are *homeomorphic* if they can be obtained from the same graph by inserting vertices into their edges.

The *chromatic number* of a graph is the number of colors that it would take to color the vertices such that no two adjacent vertices are of the same color. An independent set of a graph is made up of non-adjacent vertices.

Standard order notation will be used for the timing analysis of algorithms, see [Horowitz et al.] The reader is referred to Beineke and Wilson for any undefined terms [Beineke et al.].

2.2. Some NP-completeness Results

Lawler is credited with the first proof that the Hamiltonian circuit problem is NP-complete [Karp]. It has also been shown that the Hamiltonian circuit problem for directed graphs [Hopcroft and Ullman], the Hamiltonian path problem for undirected graphs [Horowitz et al.], and the Hamiltonian path problem for directed planar graphs [Garey, Johnson and Stockmeyer], are NP-complete. Garey, Johnson and Tarjan prove that even for the very restricted class of cubic planar 3-connected graphs the problem is NP-complete [Garey, Johnson and Tarjan].

The Hamiltonian circuit problem has been shown NP-complete for many other classes of graphs. Johnson provides the following partial list of classes of undirected graphs for which the Hamiltonian circuit problem has been shown to be NP-complete [Johnson]:

degree-k: no vertex has degree greater than k .

general planar graphs: can be drawn on a plane with no two edges intersecting.

genus-k: a graph embeddable in a surface of genus k (a plane is genus 0).

thickness-k graphs: a graph which can be partitioned into k subgraphs which are all planar.

$K_{3,3}$ -free: contains no subgraphs homeomorphic to $K_{3,3}$.

perfect: for every induced subgraph, G' , the chromatic number of G' equals the maximum clique size.

chordal: every circuit exceeding length three has an edge joining two non-consecutive vertices in the circuit.

split: chordal graphs whose complement is also chordal.

comparability: undirected graphs for which a direction can be assigned to each edge so that if $\{a,b\}$ and $\{b,c\}$ are arcs then so is $\{a,c\}$. ie. can be transformed into a transitive directed graph.

bipartite: vertices can be separated into two sets, A and B , so that all edges are from vertices in A to vertices in B .

edge: the edge graph of G has vertices corresponding to the edges of G and an edge between two vertices whenever the corresponding edges of G have one common vertex.

claw-free: does not contain $K_{1,3}$ as an induced subgraph.

bipartite planar graph with degree ≤ 3 [Itai et al.]

2.3. General Grid Graphs

Grid graphs, which are the major topic of the research, are introduced here with a look at the proof by Itai, Papadimitriou and Szwarcfter that for general grid graphs the Hamiltonian path and circuit problems are NP-complete [Itai et al.].

A grid graph is made up of vertices that lie on integer coordinates and edges that connect two vertices if the Euclidean distance between them is equal to one. If a vertex v , with coordinates (v_x, y_y) , has $v_x + v_y = 0(\text{mod } 2)$ then it is called even and otherwise it is called odd. Thus, all grid graphs are bipartite graphs since the graph is completely specified by its vertex set.

Itai et al. use reduction to prove that the Hamiltonian circuit problem for general grid graphs is NP-complete. A problem reduces to another if there is a way to solve the first problem using an algorithm that solves the second problem. This can be shown by giving a transformation of one problem into the other. Itai et al. provide a construction which transforms a bipartite planar graph, B , with degree ≤ 3 , into a grid graph that has a Hamiltonian path if and only if B has a Hamiltonian path.

The Hamiltonian circuit problem is NP-complete for planar bipartite graphs with degree ≤ 3 . Itai et al. prove this by reduction to the Hamiltonian circuit problem on planar digraphs with all vertices, v , having $\text{indegree}(v) + \text{outdegree}(v) = 3$. That problem is shown to be NP-complete

by Garey and Johnson [Garey and Johnson].

To show that the Hamiltonian path problem with specified endpoints is NP-complete, consider a grid graph with vertices of degree greater than one. Choose s to be a degree two vertex and t to be a neighbor of s . The graph has a Hamiltonian circuit if and only if there is a Hamiltonian path.

To show that the Hamiltonian path problem without specified endpoints is NP-complete, consider a grid graph without vertices of degree one and add two vertices, s and t , of degree one. There is a Hamiltonian path from s to t if and only if there is a Hamiltonian circuit.

CHAPTER 3

Sufficient Conditions for the Existence of Circuits and Paths

The purpose of this chapter is to review the classes of graphs that are always Hamiltonian or that are Hamiltonian-connected. For all these classes the known conditions are sufficient but not necessary. Thus, either it is not known whether the sufficient conditions are necessary, or graphs have been found that do not satisfy the conditions but that have Hamiltonian paths. This implies that if a graph can be recognized as belonging to one of these classes then it must be Hamiltonian (or Hamiltonian-connected) and if it does not belong to one of these classes then nothing can be said about its Hamiltonicity.

It is not always possible to construct a polynomial time algorithm which recognizes that a graph belongs to a class. Note that it is another problem to find a circuit (or path) and even if the graph can be recognized as Hamiltonian in polynomial time, a polynomial time algorithm for finding a circuit does not necessarily exist.

3.1. Hamiltonian Graphs

All the results discussed in this section show sufficient conditions for a graph to be Hamiltonian.

3.1.1. Valency Conditions

The valency of a vertex is the number of edges incident to it. Intuitively, if a graph has a lot of edges so that from each vertex there are many choices for where to go next then it is likely that the graph has a circuit.

Smith and Quinn provide a summary of four theorems connecting Hamiltonicity and valency [Quinn et al.]. They provide the 5 examples reproduced in Figure 3.1 to demonstrate that the four theorems are increasingly stringent.

The first of the theorems is known as *Dirac's theorem*. Let $v(a)$ be the valency of vertex a and let G be a connected graph with n vertices, $n \geq 3$. G is Hamiltonian if for each vertex, a , $v(a) \geq n/2$. Note in Figure 3.1(a) that $n = 6$ and the valency of each vertex is three so Dirac's condition holds. It does not hold for Figure 3.1(b) - (e).

The next theorem, *Ore's theorem*, states that if each pair of non-adjacent vertices, a and b , satisfy $v(a) + v(b) \geq n$ then G is Hamiltonian. Ore's condition is satisfied in Figure 3.1(a) and (b) but not in (c) - (e).

Let $(v_1, v_2, \dots, v_n)$ be the valences of vertices arranged in increasing order. *Posa's theorem* states that if each $k < n/2$ satisfies $v_k \geq k + 1$ then G is Hamiltonian. If n is odd then for $k = (n-1)/2$, $v_k \geq k$ and $v_{k+1} \geq k+1$ is sufficient. In Figure 3.1(c) the vector of valencies is $(2, 3, 3, 3, 3, 4)$. Since $v_1 \geq 2$ and $v_2 \geq 3$, Posa's theorem holds. It also holds in Figure 3.1(a) and

(b) but not in (d) or (e).

Finally, *Chvátal's theorem* states that for all $k < n/2$, if either $v_k \geq k + 1$ or $v_{n-k} \geq n - k$ then G is Hamiltonian. Figure 3.1(d) satisfies Chvátal's condition because the vector of valencies is $(2,2,3,4,4,5)$, $v_1 \geq 2$, and $v_2 < 3$ but $v_4 \geq 4$. Chvátal's condition also holds in Figure 3.1(a) - (c) but not (e).

Both Bixby and Wang [Bixby et al.] and Bondy and Chvátal [Bondy et al.] have developed $O(n^2)$ algorithms which find Hamiltonian circuits in graphs satisfying Chvátal's condition (and thus Dirac's, Ore's and Posa's as well).

Many other conditions of a similar nature have been shown involving valency. Ainouche has a result depending on the number of vertex disjoint paths and independent set size [Ainouche et al.]. If the graph is regular and 2-connected a theorem of Jackson's applies [Bermond]. Ghouila-Houri shows that connected digraphs with the valency of each vertex $\geq$ the number of vertices are Hamiltonian [Nash-Williams].

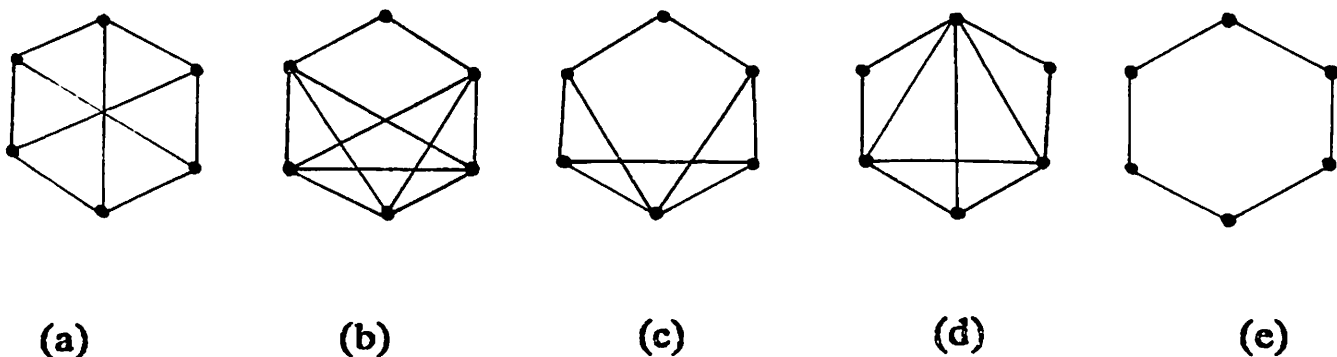


Figure 3.1. Valency conditions [Quinn et al. p. 35].

3.1.2. Connectivity Conditions

Every Hamiltonian graph must be 2-connected [Bermond] but the converse is not true. Clearly every complete graph is Hamiltonian. Determining the Hamiltonicity of graphs whose connectivity falls between these two extremes has been the subject of much study.

Whitney has shown that every 4-connected maximal planar graph is Hamiltonian [Whitney]. Based upon a simplified proof of Whitney's theorem, Asano, Kikuchi and Saito show a linear time algorithm for finding a Hamiltonian circuit in such a graph [Asano et al.].

A more general result is by Tutte; every 4-connected planar graph with at least two edges is Hamiltonian [Tutte]. Gouyou-Beauchamps gives an algorithm based on the proof which finds a circuit. The algorithm requires $O(n^3)$ time [Gouyou-Beauchamps].

Simple 3-polytopes (3-regular, 3-connected, planar graphs) have come under much investigation. Barnette has conjectured that all simple 3-polytopes whose faces are all $2n$ -gons are Hamiltonian [Goodey 1975]. Goodey has shown that this is true if the faces are either quadrilaterals or hexagons [Goodey 1975] or if they are either triangles or hexagons [Goodey 1977].

Small 3-polytopes are Hamiltonian. Okamura shows that every simple 3-polytope having 32 vertices or less is Hamiltonian [Okamura]. If the 3-

polytope is also bipartite and has less than 66 vertices then it is Hamiltonian [Holton et al.].

Interesting results have been shown for powers of graphs. The cube of a graph is one in which two vertices are joined if they are separated by at most distance three. Sekanina shows that the cube of every connected graph is Hamiltonian [Bermond]. An algorithm by Rosenstiehl requires $O(n)$ time to find a circuit in such a graph [Bermond]. The square of a graph is one in which two vertices are joined if they are separated by at most distance two. Fleischner has shown that the square of every 2-connected graph is Hamiltonian [Fleischner].

3.1.3. Subgraph Structure Conditions

Rather than considering global conditions such as valency and connectivity, the results in this section utilize knowledge of the subgraph structure to show Hamiltonicity.

Many results have been shown for graphs which do not contain $K_{1,3}$ as an induced subgraph. $K_{1,3}$ -free graphs which are locally connected are Hamiltonian [Mathews et al.]. Mathews and Sumner [Mathews et al.] and Gould and Jacobson [Gould et al.] give further results along this line.

An edge graph (or line graph), $L(G)$, of a graph (G) has vertices corresponding to the edges of G and an edge whenever the corresponding edges in G are adjacent. Goodman and Hedetniemi show that edge graphs of

Eulerian graphs and edge graphs of Hamiltonian graphs are Hamiltonian if they can be partitioned into subgraphs with particular characteristics [Goodman et al.].

3.1.4. Other

There are many other results on Hamiltonicity for graphs which fit into none of the above mentioned categories.

A graph is vertex-transitive if for any two vertices there is an automorphism from one to the other. Lovasz has conjectured that every connected, vertex-transitive graph has a Hamiltonian path. This has been verified for all graphs with $2p$, $3p$, $4p$, $5p$, p^2 and p^3 vertices for p a prime [Marusic et al.]. Further results can be found in [Lipman], [Marusic et al.] and [Gallian et al.].

A Halin graph is formed from a tree with no degree-2 vertices whose leaves are connected by a circuit that crosses no edge. Halin graphs are Hamiltonian [Johnson]. Every tournament (directed, complete graph), T_n with n vertices where $n = 2k \geq 28$ has an antidirected Hamiltonian dicircuit (a circuit in which the direction of consecutive edges alternates) [Rosenfeld]. A grid graph is a graph whose vertices lie on integer coordinates and edges connect vertices separated by a distance of one. A grid graph is rectangular if its vertices are specified by $V = \{v: 1 \leq v_x \leq m \text{ and } 1 \leq v_y \leq n\}$ where each vertex, v , has coordinates (v_x, v_y) . Every rectangular grid graph with an even number of vertices has a Hamiltonian circuit [Luccio et al.]. Bermond

[Bermond] and Nash-Williams [Nash-Williams] give further results on Hamiltonicity.

3.2. Hamiltonian-Connected Graphs

There have been far fewer results concerning Hamiltonian-connected graphs. A few are mentioned here.

Plummer has conjectured that every 4-connected planar graph is Hamiltonian-connected [Plummer]. This has been shown in [Thomassen] who also gives a polynomial time algorithm for finding the paths.

Some m -regular graphs have been shown to be Hamiltonian-connected, see [Tomescu]. Also, the cube of every connected graph is Hamiltonian-connected [Bermond]. For further results see [Naddef et al.], [Bermond] and [Nash-Williams].

CHAPTER 4

Polynomial Time Algorithms

This chapter looks at restricted classes of graphs which permit polynomial time algorithms for the Hamiltonian circuit/path problem. These problems are different from those in the previous section because for these graphs it is possible to present both necessary and sufficient conditions for the existence of a path or circuit. The algorithms, developed from these conditions, also find a path or circuit if one exists. All algorithms are examined briefly except that of rectangular grid graphs which is presented in more detail as it relates closely to the research.

4.1. k -Trees and Series-Parallel Graphs

A *tree* is an undirected connected graph without any circuits. Clearly no tree has a Hamiltonian circuit and a tree has a Hamiltonian path if it is a path.

A more general construct closely related to a tree is a *partial k -tree*. A k -tree is: “(1) A complete graph on k vertices is a k -tree. (2) If $G = (V, E)$ is a k -tree and $V' \subseteq V$ is a set of k vertices that induces a complete subgraph in G , then the graph obtained by adding a new vertex v to V together with an edge from v to every vertex in V' is also a k -tree [See Figure 4.1.] A graph

is a partial k -tree if there is some k -tree G' of which it is a subgraph." [Johnson p. 439]. If k is fixed then a polynomial time algorithm can find a Hamiltonian circuit in a partial k -tree if there is one [Johnson]. However, the algorithm grows exponentially with k .

A *series-parallel graph* is: (1) an edge from a vertex s to a vertex t , (2) a series connection between two series-parallel graphs, SP_1 , SP_2 (ie. replace t in SP_1 and s in SP_2 with a single vertex), (3) a parallel connection between two series-parallel graphs (ie. replace s in SP_1 and s in SP_2 with a single vertex; similarly for t), see Figure 4.2. Wald and Colbourn have shown that all series-parallel graphs are partial 2-trees [Wald et al.]. Thus the algorithms for partial 2-trees apply here as well.

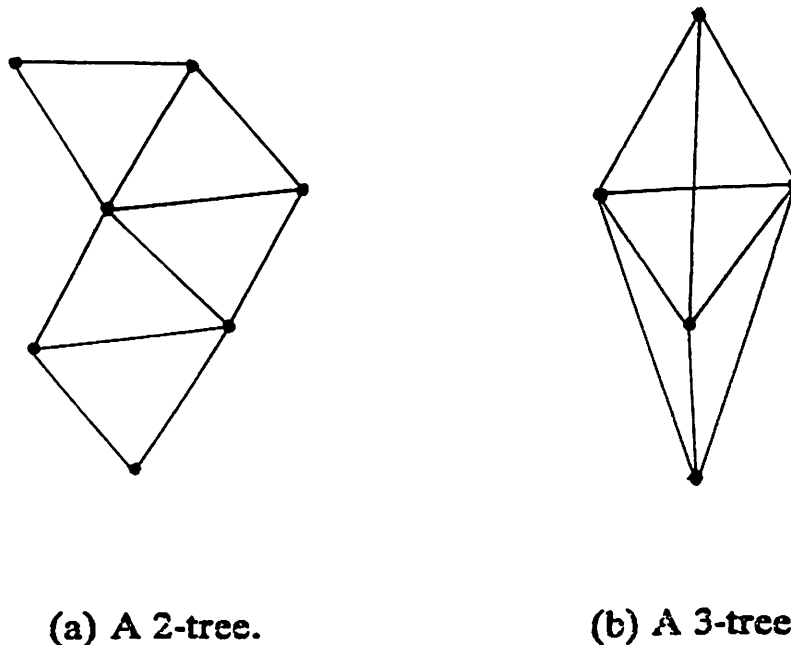


Figure 4.1. k -trees.

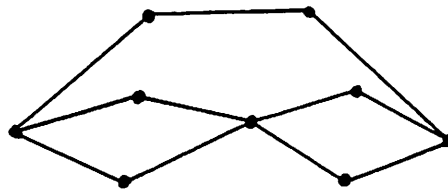


Figure 4.2. A series-parallel graph.

4.2. Outerplanar Graphs

An *outerplanar* graph can be drawn in the plane so that all the vertices lie on the same face, see Figure 4.3. Outerplanar graphs are all series-parallel graphs so the Hamiltonian circuit problem is polynomial for them.

Outerplanar graphs, also called 1-outerplanar graphs, are a subclass of *k-outerplanar graphs*. "A graph is *k-outerplanar* [$k \geq 2$] if it has a planar embedding such that if all vertices on the exterior face are deleted, the connected components of the remaining graph are all $(k - 1)$ -outerplanar" [Johnson, p.442]. Again a polynomial time algorithm depends on a fixed k .

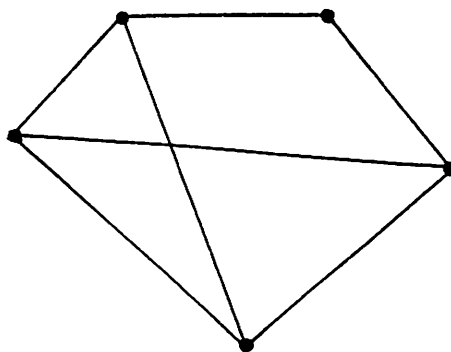


Figure 4.3. An outerplanar graph.

Dynamic programming is the algorithmic technique used by Baker [Baker].

4.3. Cographs

Corneil, Lerchs and Burlingham give a polynomial time algorithm for determining if a cograph is Hamiltonian or Hamiltonian connected [Corneil et al.]. A *cograph*, or complement reducible graph, is a graph with no induced subgraph isomorphic to P_4 . A cograph is shown in Figure 4.4.

Corneil states the following necessary and sufficient conditions for a cograph, $G = (V, E)$:

- 1) G is Hamiltonian iff $S(G) \leq 0$ and $|V| \geq 3$
- 2) G is Hamiltonian connected iff $S(G) < 0$

$S(G)$ is the *scattering number* of G . $S(G) = \max(c(G - S) - |S|)$ where $S \subseteq V$ and $c(G - S)$, the number of components of $G - S$, does not equal one. In general, the larger the scattering number the less well connected the graph. Notice that the scattering number of the graph in Figure 4.4 is 1 (removing a and b leaves three components) so there is no circuit.

A cograph has the interesting property of being uniquely representable by a tree. This tree, called a *cotree*, is useful for solving many problems on cographs including finding the scattering number, see [Corneil et al.].

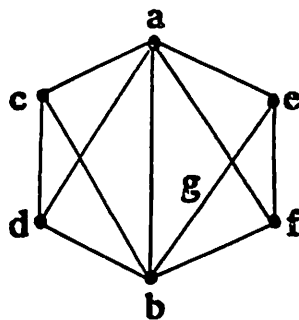


Figure 4.4. A cograph [Corneil et al. p. 166].

4.4. Striped Graphs

Let $I = \{0, 1, \dots, n-1\}$ be an index set for the vertices of a graph. A stripe can be defined for each $k \in I$. The k th stripe in a graph with n vertices is defined as follows: $R_k = \{\{i, [i+k]_n\} \mid i \in I\}$, where $[a]_n$ means a modulo n . A graph is *striped* if its edges form strips. Figure 4.5(a) shows a graph with one stripe, R_3 .

Necessary and sufficient conditions for a graph with one stripe to be Hamiltonian have been shown. A graph with the k th stripe is Hamiltonian iff $\gcd(k, n) = 1$ where $\gcd$ is the greatest common divisor [Garfinkel et al.]. Note that in Figure 4.5(a), $\gcd(3, 5) = 1$ and there is a circuit: 1-3-0-2-4-1.

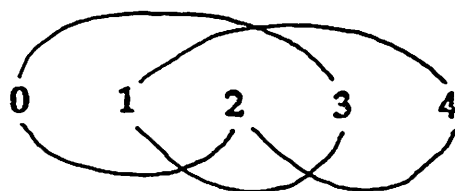
Clearly if a graph with two stripes satisfies $\gcd(k, n) = 1$ for either stripe then it is Hamiltonian. However, a circuit may also be formed by including some edges from both stripes. Garfinkel and Sundararaghavan give neces-

sary and sufficient conditions for two-stripe graphs to have a circuit which contains edges from both stripes. Let p and q be the two stripes and $G(p,q)$ be the graph containing stripes p and q . Define $g = \gcd(p-q,n)$. $G(p,q)$ is Hamiltonian iff $\gcd(p,q,n) = 1$, $g \geq 2$, and there exists an $m \in [1, g-1]$ such that $\gcd(m^*, n) = g$ where $m^* = pm + q(g-m)$ [Garfinkel et al.].

An algorithm not worse than $O(n \log n)$ has been devised to find the Hamiltonian circuits in $G(p,q)$. A *binary symmetric circular permutation* is found based on a valid m . The permutation has two values (hence it is a binary permutation), p and q , arranged into n/g identical groups of g elements, m of which equal p and $g-m$ of which equal q . The g elements are arranged in a circle and all permutations obtained by starting from any position in that circle and moving around n/g times are considered equivalent. Each permutation defines a Hamiltonian circuit in $G(p,q)$; starting with any vertex, the next vertex is obtained from R_p if the next element of the permutation is p and from R_q if the next element of the permutation is q . Consider the example in Figure 4.5(b). Here $n = 12$, $p = 7$, and $q = 3$. $g = 4$. $m = 1$ is valid which leads to the binary symmetric circular permutation 7333 7333 7333. The circuit given by this is 1-8-11-2-5-12-3-6-9-4-7-10-1.

4.5. Proper Interval Graphs

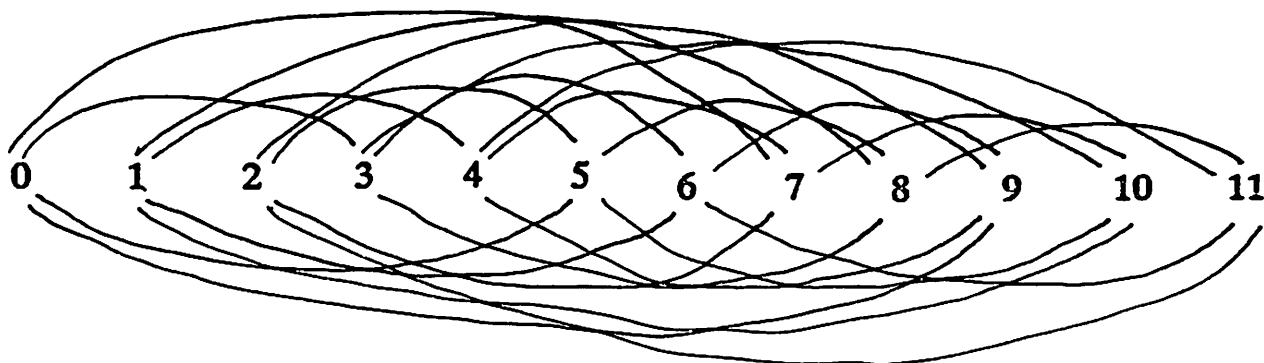
Given F , a family of intervals on the real line, an *interval graph*, G is constructed where the vertices are the intervals and an edge exists between



$$n = 5, I = \{0,1,2,3,4\}$$

$$R_3 = \{\{0,3\}, \{1,4\}, \{2,0\}, \{3,1\}, \{4,2\}\}$$

(a) A one-stripe graph.



(b) A two-stripe graph.

Figure 4.5. Striped graphs.

two vertices if the corresponding intervals intersect. The set F is called the interval model for G . An interval graph is called *proper* if the set F is made up of intervals of unit length. Bertossi has shown how the Hamiltonian circuit problem can be solved in $O(n \log n)$ time for proper interval graphs [Bertossi]. His method is outlined below.

Figure 4.6 shows an interval model, $I = \{I_1, \dots, I_n\}$ with $I_i = [a_i, b_i]$, and the corresponding proper interval graph. The interval with the smallest

a_i is called *First* and the interval with the largest b_i is called *Last*. The following three lemmas that Bertossi proves can be seen easily by looking at the graph:

Lemma 1. A proper interval graph is connected if and only if the union from $i = 1$ to n of $[a_i, b_i]$ forms a single interval.

Lemma 2. A proper interval graph has a Hamiltonian path if and only if it is connected.

Lemma 3. A proper interval graph has a Hamiltonian circuit if and only if there exist two paths which intersect only at First and Last.

The algorithm finds these two paths by a scan through the ordered list of intervals.

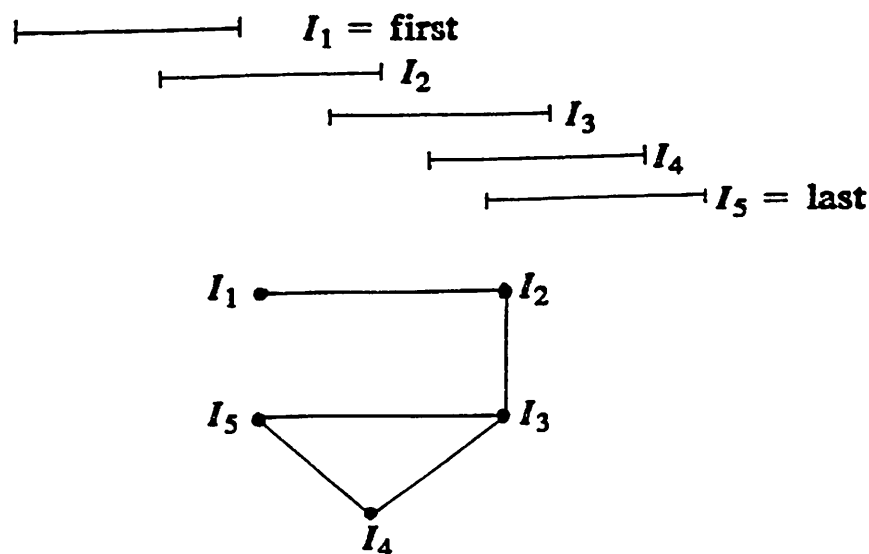


Figure 4.6. A proper interval graph.

4.6. Interval Graphs

A linear time algorithm for the Hamiltonian circuit problem in non-proper interval graphs has been found by Keil [Keil]. The algorithm is summarized here.

In an interval graph, G , a clique, c , corresponds to a set of mutually overlapping intervals. Note that each clique can be represented by a point in the intersection of the intervals forming the clique. The set of cliques, $c_1, \dots, c_n$, can then be ordered by their representative points, see Figure 4.7.

A *conductor* for a clique, c_i , is a vertex which appears in c_i and c_{i+1} . So, for example, the vertex corresponding to interval b in Figure 4.7 is a conductor for c_1 as it appears in c_1 and c_2 . A *spanning path* for cliques $L(c_i) = \{c_1, \dots, c_i\}$ is one which covers all of the vertices which appear only in $L(c_i)$ and which has two conductors of c_i as endpoints. A path is straight if there is a point embedding which assigns a point, $q(v_i)$ from R_i , the set of representative points of the cliques containing vertex v_i , to v_i such that $q(v_i) \in R_{i+1}$ and $q(v_r) \leq q(v_{r+1})$. A path can always be made straight. A *U-shaped spanning path* from A to B has a vertex $P \in c_1$ such that there is a straight path from A to P and a straight path from P to B .

An interval graph with m cliques has a Hamiltonian circuit if and only if each $L(c_i)$ has a U-shaped spanning path. The algorithm visits each clique

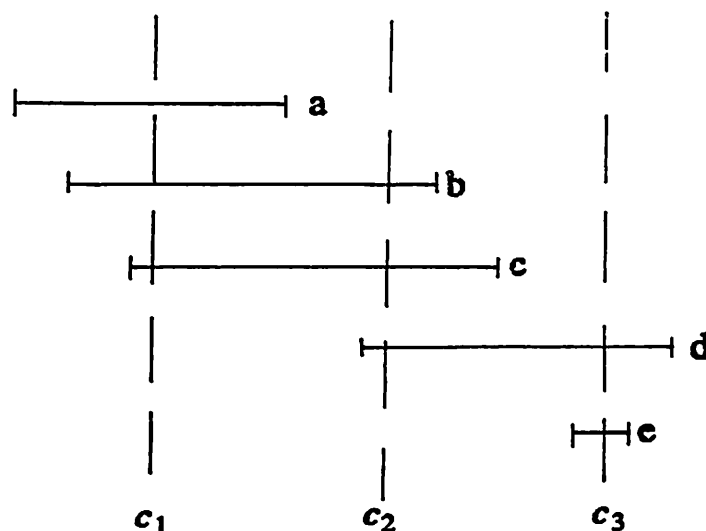


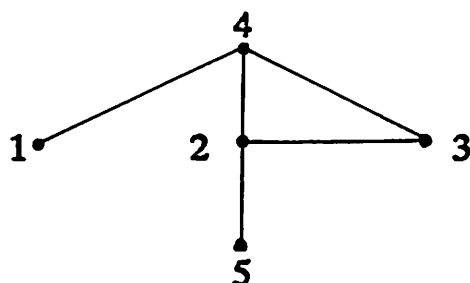
Figure 4.7. The representative points of cliques in an interval model.

building up a larger U-shaped spanning path until the last clique is reached and the endpoints are joined to form a circuit.

4.7. Bipartite Permutation Graphs

The restriction to permutation graphs has led to polynomial time algorithms for many problems which are NP-complete for general graphs. Brandstadt and Kratsch show that with a further restriction to bipartite permutation graphs the Hamiltonian circuit problem can be solved in $O(n)$ time [Brandstadt et al.].

Let γ_n be the set of all permutations of $\{1, \dots, n\}$. $\pi \in \gamma_n$ defines a *permutation graph* in which the vertices are $\{1, \dots, n\}$ and an edge $\{i, j\}$ exists if $j > i$ and is to the left of i in π . Figure 4.8 shows a permutation and the corresponding permutation graph.



$$\pi = \{4, 1, 3, 5, 2\}$$

Figure 4.8. A permutation graph.

A *bipartite permutation graph* has been shown to have the following property: if one set of vertices, X , is lined up on one side of the unit square, the other set Y , is lined up in the opposite side, $\{u, v\}$ and $\{x, y\}$ are crossing edges for $u, x \in X$ and $v, y \in Y$ then the edges $\{u, y\}$ and $\{x, v\}$ must exist, see Figure 4.9.

Notice that the increasing subsequences in π form an independent set. A *monotone increasing partition*, or MIP, which is a partition of the set into increasing subsequences, can be formed for any permutation graph using an

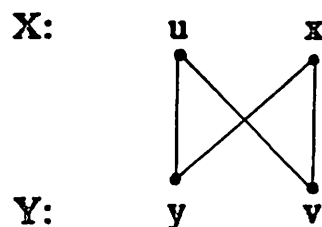


Figure 4.9. Characterization of a bipartite permutation graph.

algorithm in [Brandstädt et al.]. Notice that for bipartite permutation graphs the MIP is composed of two sets, S_1 and S_2 .

The following theorem [Brandstädt et al. p. 11] specifies the sufficient and necessary condition for finding Hamiltonian circuits in bipartite permutation graphs:

Theorem: A bipartite permutation graph has a Hamiltonian circuit if and only if for each pair of neighbors in the MIP, $a_i, a_{i+1} \in S_1$, $b_i, b_{i+1} \in S_2$, the edges $\{a_i, b_i\}$, $\{a_i, b_{i+1}\}$, $\{a_{i+1}, b_i\}$, and $\{a_{i+1}, b_{i+1}\}$ exist, see Figure 4.10.

The algorithm for finding the circuit is straightforward. All edges $\{a_i, b_i\}$, $1 < i < n$ are omitted. Clearly the algorithm runs in $O(n)$ time.

4.8. Rectangular Grid Graphs

This section of the thesis summarizes the work done by Itai, Papadimitriou, and Szwarcfiter on rectangular grid graphs in [Itai et al.]. It should be noted that Luccio and Mugnai first studied rectangular grid graphs and presented some of the results mentioned here as unproven facts [Luccio et

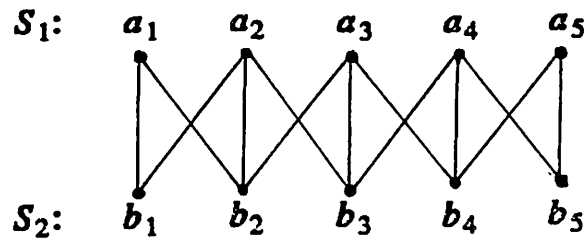


Figure 4.10. Edges between MIP neighbors.

al.].

A rectangular grid graph, $R(m,n)$, is specified by a vertex set $V = \{v: 1 \leq v_x \leq m \text{ and } 1 \leq v_y \leq n\}$ where each vertex, v , has coordinates (v_x, v_y) . m and n are called the dimensions of R . (R,s,t) is the notation for the Hamiltonian path problem in R for specified endpoints, s and t . Three necessary conditions are developed and shown to be sufficient to solve (R,s,t) .

Recall that grid graphs are bipartite. The set of vertices can be separated into two sets, V_1 and V_2 . Suppose V_1 and V_2 are colored by the colors, black and white, respectively. *Color compatibility*, the first necessary condition, is then defined by the following:

Definition: The Hamiltonian path problem (R,s,t) is color compatible if:

1. $(|V_1| = |V_2|)$ and s and t have different color, or
2. $(|V_1| = |V_2| + 1)$ and $s, t \in V_1$.

The second necessary condition, which also applies to graphs other than grid graphs, is *connectivity*. If any of $G - \{s\}$, $G - \{t\}$, or $G - \{s,t\}$ are not connected then there is no Hamiltonian path from s to t in G . A problem which satisfies the first two necessary conditions, color compatibility and connectivity, is called *acceptable*. A problem is *forbidden* if:

(G,s,t) is isomorphic to (G',s',t') which satisfies:

1. $G' = R(m,n)$ with $n = 3$ and m even.
2. s' is colored differently from t' and the left corners of G' .

3. $s'_x < t'_x - 1$ or $s'_y = 2$ and $s'_x < t'_x$.

The third necessary condition is that the problem is not forbidden. Figure 4.11 shows examples of problems not satisfying the necessary conditions. Figure 4.11(a) shows a problem which is not color compatible, (b) shows a problem in which $G - \{t\}$ is disconnected and in which $G - \{s, t\}$ is disconnected and the problems in (c) shows problems which are forbidden.

A problem is called *solvable* if there is a Hamiltonian path from s to t in R . A proof is given showing that any acceptable problem which is not forbidden is solvable. The method of proof is to show that any problem can be broken into smaller solvable problems by two methods, *stripping* and *splitting*, and that the solutions to the smaller problems can be combined to form a solution for the original problem. There is a set of eight problems, called

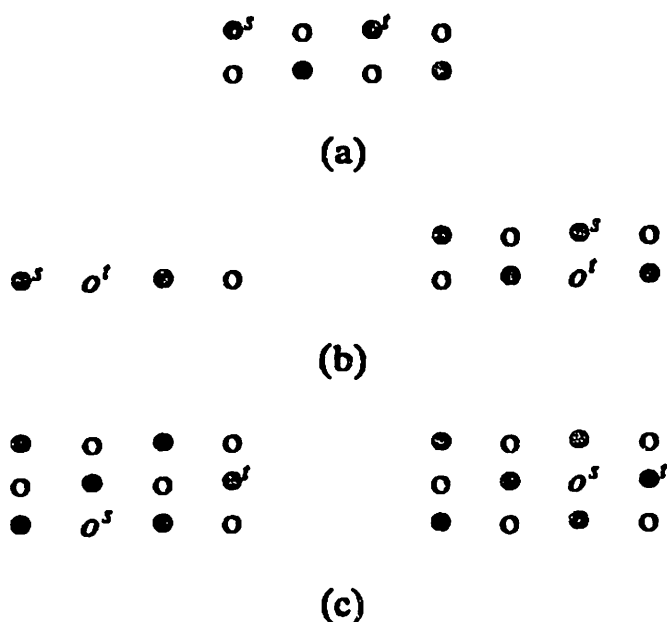


Figure 4.11. Problems which don't satisfy the necessary conditions.

prime problems, which cannot be stripped or split and their solutions are given.

The following definition of stripping is given in [Itai et al.]:

Definition: A *separation* of a rectangle R is a partition of R into two subrectangles, i.e., $V(R)$ is a disjoint union of $V(R_1)$ and $V(R_2)$. A strip (subrectangle) S *strips* a Hamiltonian path problem (R, s, t) if

1. $S, R-S$ is a separation of R ,
2. $s, t \in R-S$, and
3. $(R-S, s, t)$ is acceptable and not forbidden [Itai et al. p. 682].

A proof by construction is given to show that if (R, s, t) is acceptable and not forbidden and $(R-S, s, t)$ has a solution, then (R, s, t) has a solution. See Figure 4.12 for an example of the construction required. Note that if S has one row or column then the construction is not possible. Thus the definition of a strip must be amended such that a strip is a subrectangle with more than one row and column.

The following definition of splitting is given in [Itai et al.]:

Let (R, s, t) be an acceptable Hamiltonian path problem which is not forbidden and pq an edge. pq *splits* (R, s, t) if there exists a separation of R into R_p and R_q such that :

1. $s, p \in R_p$ and (R_p, s, p) is acceptable and not forbidden, and
2. $q, t \in R_q$ and (R_q, q, t) is acceptable and not forbidden [Itai et al. p. 683].

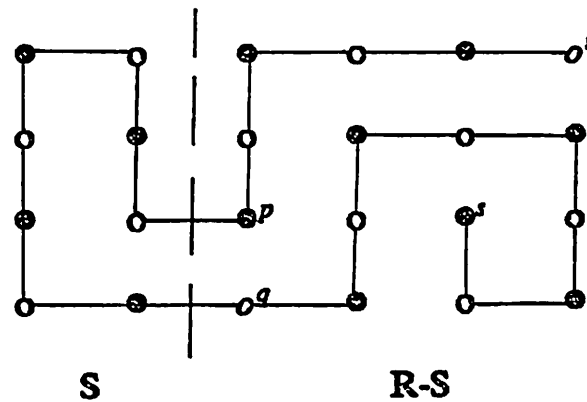


Figure 4.12. Stripping.

It is clear from the definition of splitting that when (R_p, s, p) and (R_q, q, t) have solutions then (R, s, t) has a solution. Figure 4.13 shows an example of a split.

A problem is *prime* if it cannot be stripped or split. Figure 4.14 shows the eight prime problems. For all larger problems either a split or a strip is possible.

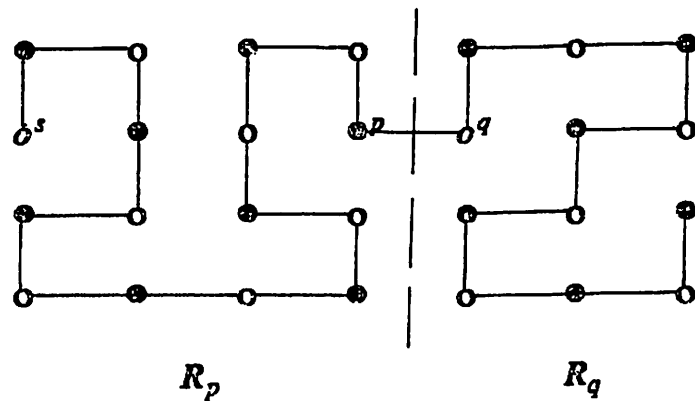


Figure 4.13. Splitting.

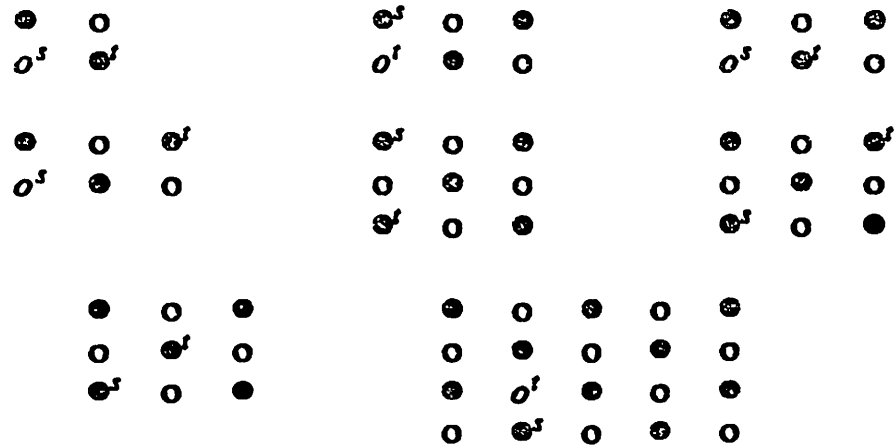


Figure 4.14. The prime problems.

To determine whether (R,s,t) is Hamiltonian, first determine whether it is acceptable and not forbidden. If it is not acceptable or it is forbidden then report that (R,s,t) is not Hamiltonian. Otherwise, divide the problem into a number of prime problems by splitting and stripping into smaller and smaller acceptable problems which are not forbidden. Solve each prime problem and merge the solutions using the constructions shown in Figures 4.12 and 4.13.

CHAPTER 5

Non-rectangular Grid Graphs: A Solution Strategy

As is indicated in earlier sections of this thesis the Hamiltonian path problem with specified endpoints for general grid graphs is NP-complete and the problem for rectangular grid graphs can be solved in linear time. The problem for non-rectangular grid graphs is open. The remainder of this thesis considers this problem. A solution strategy is presented. Examples are used to show where the difficulties arise. These serve to illustrate how various restrictions can be used to remove these difficulties. This leads to a series of restricted grid graph problems. Chapters Six through Eight show that these can be solved in linear time.

5.1. A Solution Strategy

A *grid graph* is a graph whose vertices lie on integer coordinates and whose edges connect vertices separated by distance one. Since grid graphs are bipartite the vertex set, $V = (V_1 \cup V_2)$. V_1 is denoted by $\bullet$ and V_2 is denoted by $\circ$. A grid graph is said to have holes if any of its interior faces has area greater than one.

A *rectilinear polygon* is a polygon whose sides are all either vertical or horizontal. A *rectilinear region* is a rectilinear polygon with rectilinear polyg-

onal holes cut out. Note that a grid graph can be completely specified by a rectilinear region, see Figure 5.1. Consider the following solution strategy for finding Hamiltonian paths in non-rectangular grid graphs:

Step 1. Decompose the region corresponding to the grid graph into rectangles, see Figure 5.1. A *decomposition* of a region is a separation into non-overlapping pieces. Note that the corresponding grid graph becomes separated into rectangular grid graphs.

Step 2. Find a connected path of rectangles from the rectangle containing s , the starting vertex, to the rectangle containing t , the ending vertex. Two rectangles are connected if they are adjacent. In Figure 5.2 such a path is A-B-C.

Step 3. Strip off all of the rectangles not on the path. Strip has the same meaning here as in the rectangular grid graph algorithm of Itai et al. In Figure 5.2, strip off D, E and F. The result is shown in Figure 5.3.

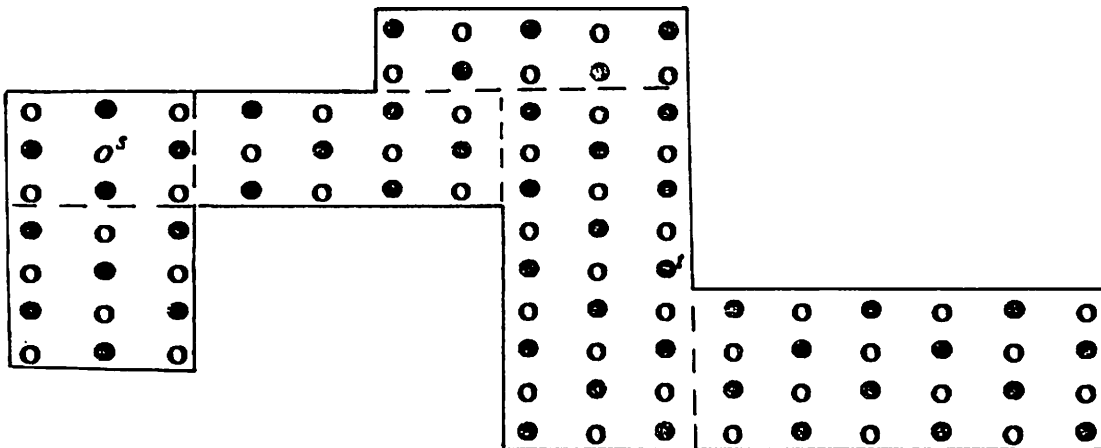


Figure 5.1. A decomposed grid graph.

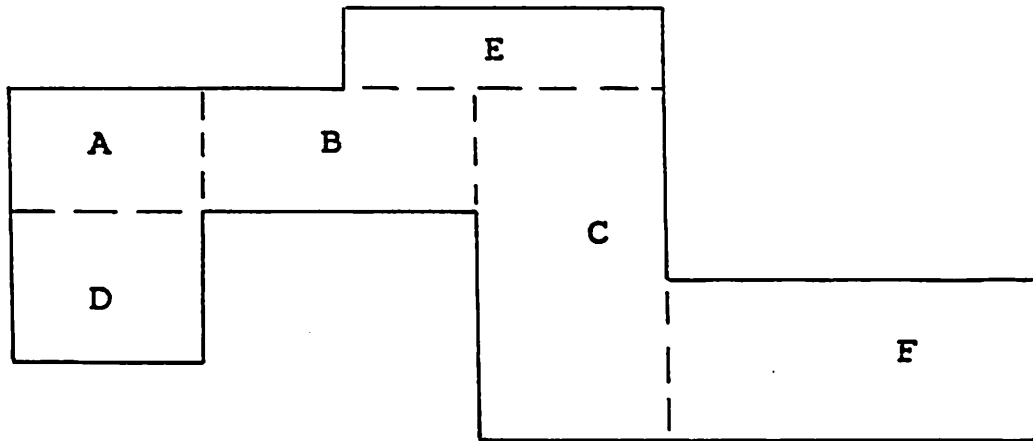


Figure 5.2. A connected path of rectangles.

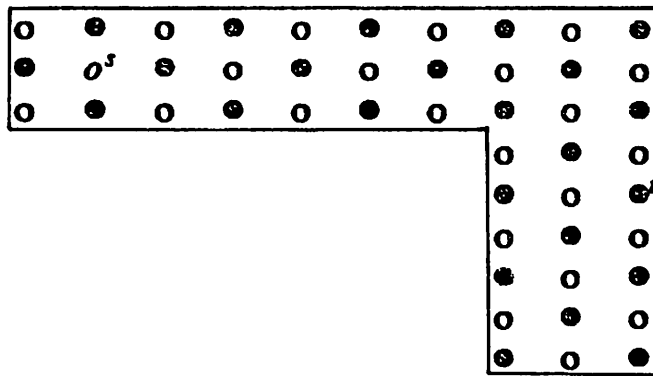


Figure 5.3. Stripping extra rectangles.

Step 4. Solve the remaining problem by splitting (in the Itai et al. sense). If the problem cannot be solved then the original problem had no path. Splitting should occur along the rectangle boundaries. This leaves rectangular problems which can be solved using the Itai et al. algorithm, see Figure 5.4.

Step 5. Solve the stripped portions and add them into the solution. Figure 5.5 shows the final path. Some of the difficulties with this approach will

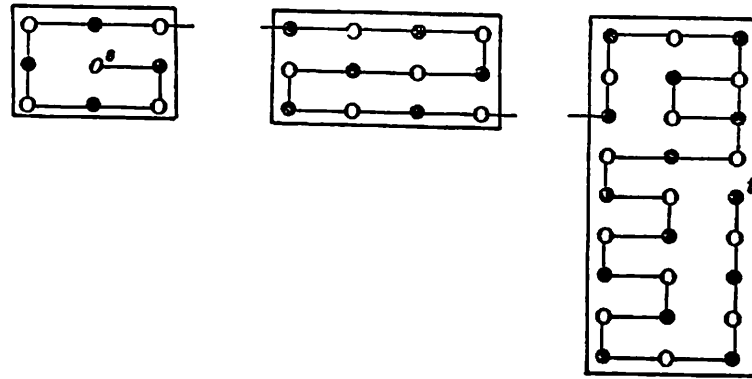


Figure 5.4. Splitting the problem.

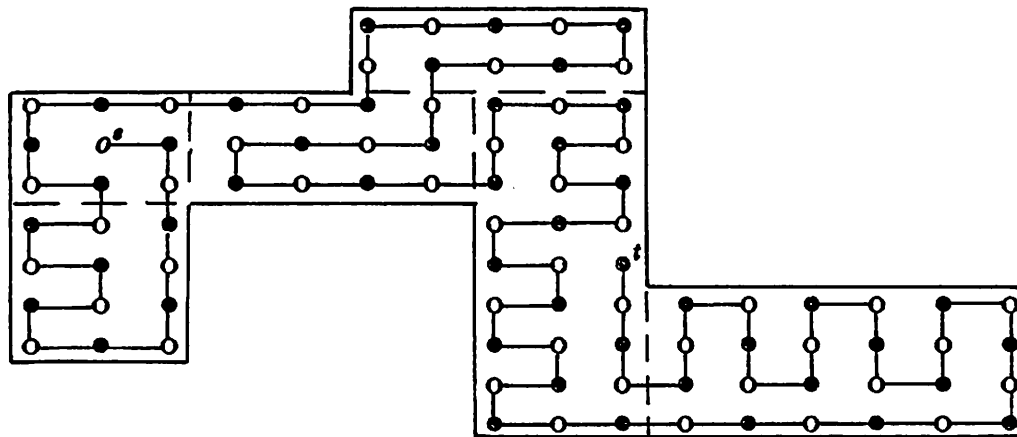


Figure 5.5. The final solution.

now be discussed.

5.1.1. The Decomposition

The first problem is the decomposition of a rectilinear region. Any one of a number of decomposition algorithms can be applied and many decompositions are possible, see [Keil and Sack]. Notice that for the decomposition shown in Figure 5.6(a) the solution strategy will correctly find a path but for

that shown in Figure 5.6(b) the solution strategy cannot find a path. One possible solution to this problem is to relax the assumption made in Step 4, that if the stripped problem cannot be solved then there is no path, by considering all possible decompositions before concluding that there is no path. Since there appears to be a large number of possible decompositions it does not seem feasible to explore all of them.

5.1.2. Finding a Path of Rectangles

Step two requires finding a path of rectangles from s to t . There may be many such rectangle paths but it is necessary to find only one. If the decomposed problem is modelled as a graph with vertices representing the rectangles and edges between adjacent rectangles then this problem reduces to finding a path between two vertices in a graph, see Figure 5.7. Since the graph is connected this should not pose any difficulties.

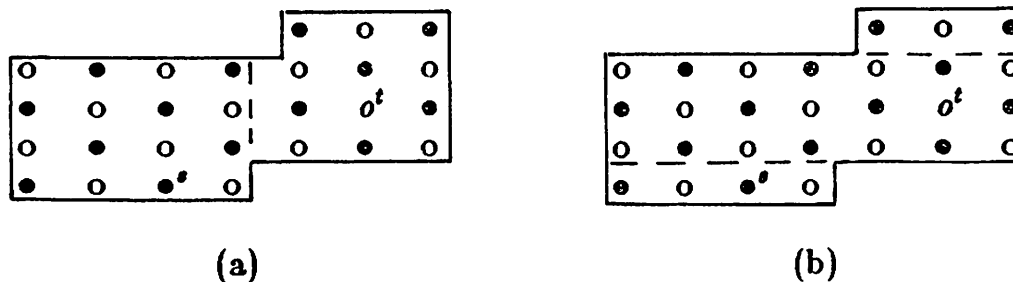


Figure 5.6. Two decompositions of the same graph.

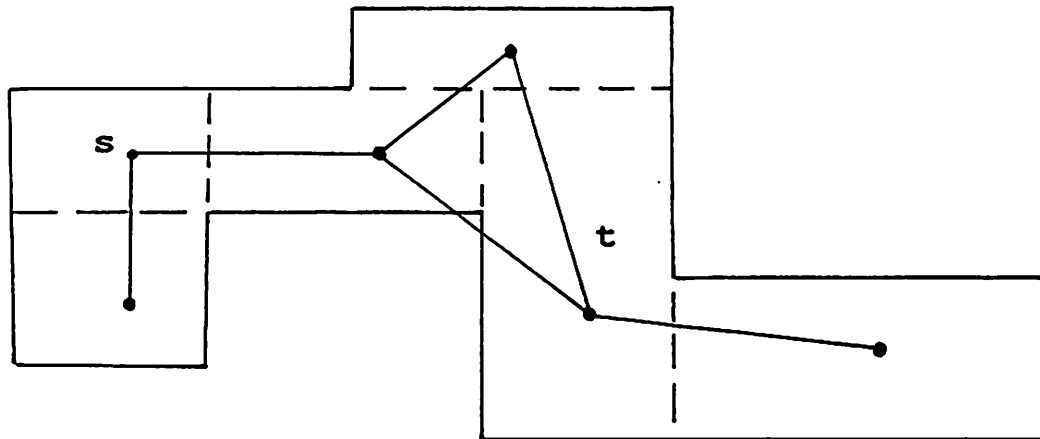


Figure 5.7. Finding a path of rectangles.

5.1.3. Stripping Rectangles Not on the Path

Consider now the problem of stripping the rectangles which are not on the path. Recall the stripping operation from the work of Itai et al. A strip, S , consists of at least two rows or columns containing an even number of vertices and not containing s or t . S is stripped from the problem and later added back into the solution through the vertices p and q . p and q are chosen such that $\{p, q\}$ is an edge on the solution of the stripped problem, p' and q' are adjacent to p and q in S and (S, p', q') is acceptable and not forbidden. Note that because $\{p, q\}$ is an edge, p and q must be side by side. An equivalent formulation of the stripping problem is to strip S , add an extra edge, $\{p, q\}$, to the stripped portion of the rectangle and then solve the stripped portion insisting that $\{p, q\}$ be included in the path, see Figure 5.8.

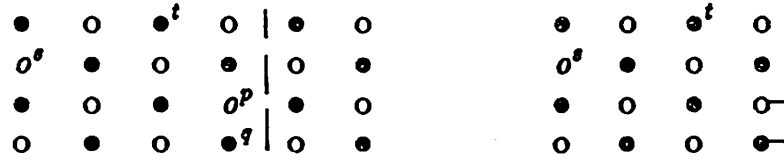


Figure 5.8. Two equivalent problems.

Consider stripping a rectangle, S , with one row or column as shown in Figure 5.9. Clearly there is no possible choice of $\{p, q\}$. Thus the rectangle cannot be stripped. However, note that such a problem has no solution.

Suppose it is necessary to remove a rectangle containing an odd number of vertices. In this case, in order for (S, p', q') to be color compatible, p' and q' must be of the same color (and thus cannot be side by side) so the solution to the stripped problem cannot contain edge $\{p, q\}$. However, an edge can be added as shown in Figure 5.10 and a path found which goes through the edge. Notice that the definition of color compatibility must be amended to account for the extra edges. If the color of the vertices in V_1 is 1 and the color of the vertices in V_2 is -1 then the problem is color compatible if the

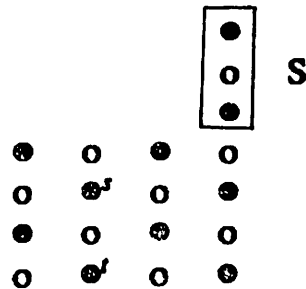


Figure 5.9. Stripping a rectangle with one row or column.

sum of the colors of the endpoints of the extra edges, the color of s and the color of t equals 0 if $|V_1| = |V_2|$ and 2 if $|V_1| = |V_2| + 1$.

So far it has been shown how to strip off rectangles with either an even or an odd number of vertices by adding an extra edge. In general, however, we will need to strip off larger chunks than just rectangles. Consider Figure 5.11. It is impossible to strip portion A because there is no choice of $\{p, q\}$. However, note that the problem has no solution. Consider Figure 5.12. A is the portion to be stripped off. Notice that A contains two more $\bullet$'s than $\circ$'s. Adding one extra edge would leave a problem which was not color compati-

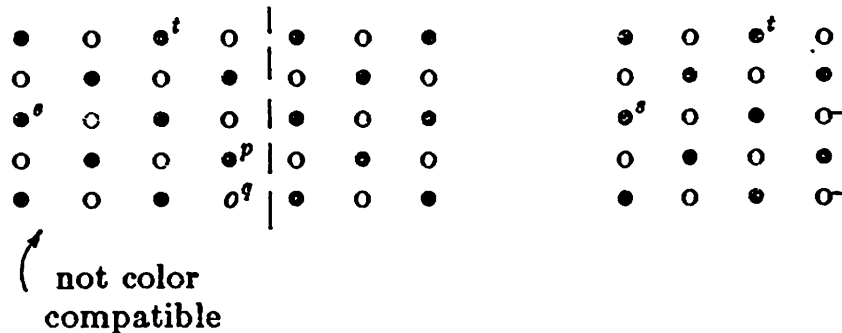


Figure 5.10. Stripping an odd number of vertices.

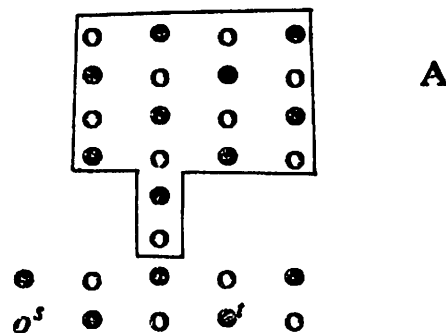


Figure 5.11. Stripping a chunk connected by a single row or column.

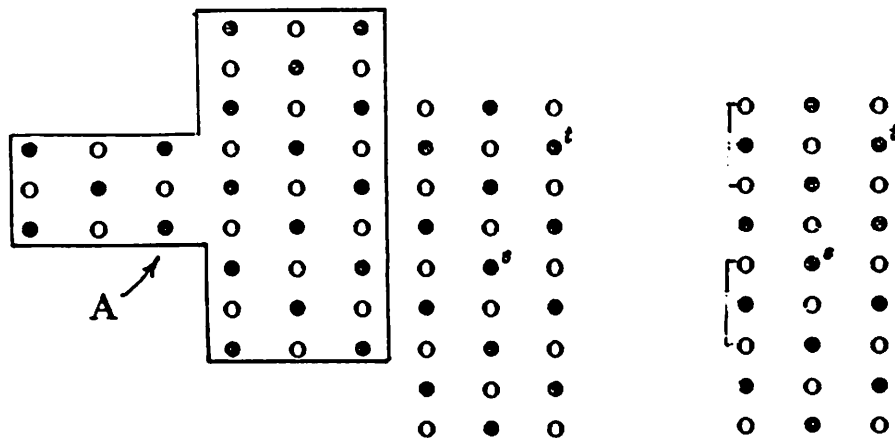


Figure 5.12. Removing big chunks.

ble. In this case it is necessary to add two extra edges. In general, many extra edges may be required. This leads to another set of problems for which no path can be found. If the number of vertices of each color in the set of extra edges exceeds the number of vertices of that color on the boundary between A and the remainder of the graph then it appears that there can be no path. Figure 5.13 shows at least one graph where this is the case.

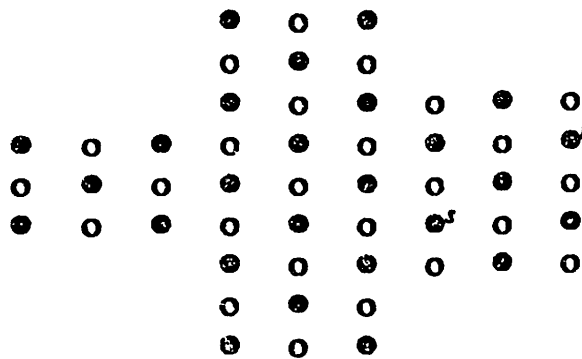


Figure 5.13. No path.

5.1.4. Solving the Stripped Problem

In Step four the stripped problem is solved. Consider the stripped problem, with extra edges added, shown in Figure 5.14(a). To solve this, continually cover a rectangle and move into the next one. This entails solving small problems, an example of which is shown in Figure 5.14(b), where it is necessary to find a path starting at s , going through all the extra edges and ending at some p on some side other than the one s is on (if s is on a side).

Consider the problem in Figure 5.15(a). To solve the first small rectangle problem, shown in Figure 5.15(b), it is necessary to add an extra edge in order to maintain color compatibility. If there is no room to add an extra edge, as is the case for the first small rectangle in the problem shown in Figure 5.15(c) then the problem cannot be solved by the solution strategy.

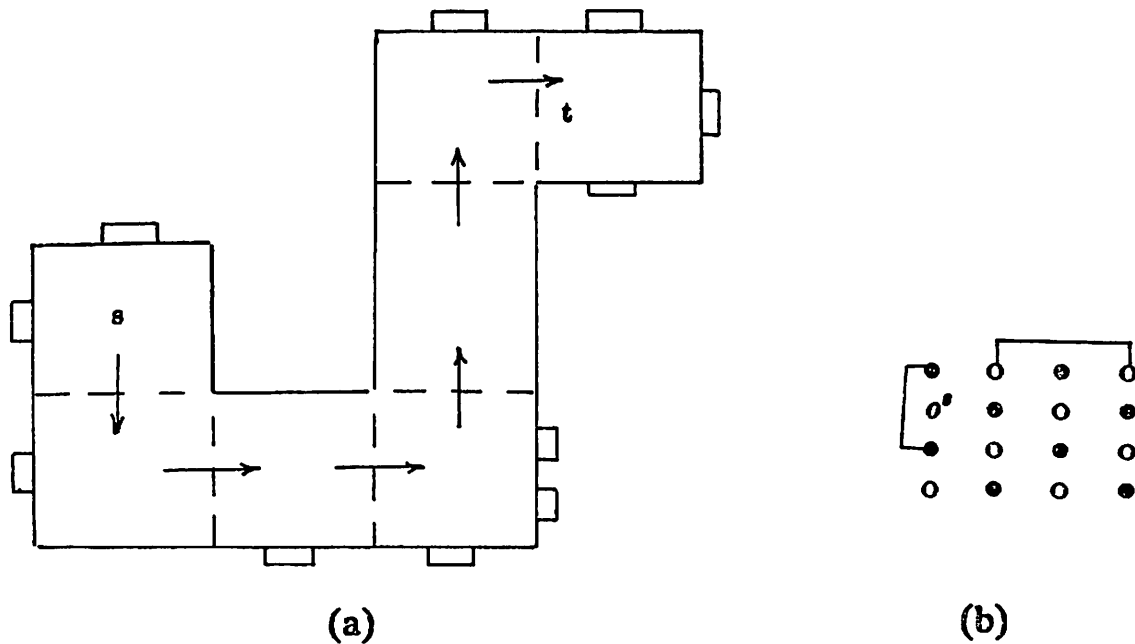


Figure 5.14. Splitting.

Figure 5.15(d) shows a problem in which the required extra edges can be added but in which the resulting small rectangle problem cannot be solved.

5.1.5. Solving the Extraneous Portions

The final step is to solve the portions which were stripped off. These problems turn out to be smaller problems of the original form so no additional difficulties should occur.

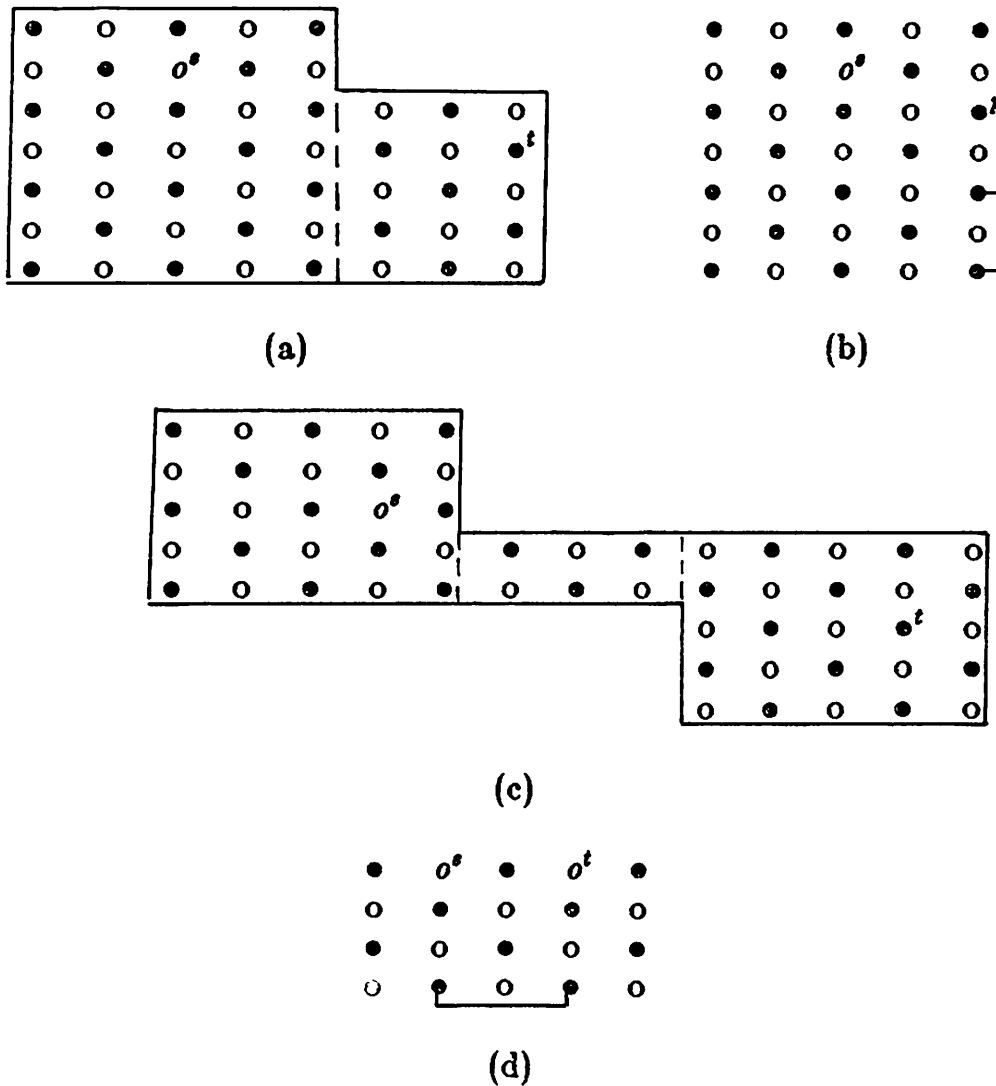


Figure 5.15. Difficulties in stripping large chunks.

5.2. Useful Restrictions

This section describes some restricted graphs for which the difficulties discussed in the previous section either disappear or appear to be manageable. First the decomposition will be defined. A *concave vertex* of a rectilinear polygon is one in which the exterior angle formed by the two edges meeting at the vertex is a right angle. All of the other vertices are convex. A *concave-vertex decomposition* of a rectilinear region is formed by constructing horizontal and vertical lines through each concave vertex of the rectilinear polygon representing the boundary of the grid graph and through each convex vertex of the rectilinear polygons forming the holes. Figure 5.16 shows the concave-vertex decomposition of a polygon. This appears to be a useful decomposition because it is unique for each region. The *size* of a concave-vertex decomposition is the dimension of the smallest rectangle formed. Fig-

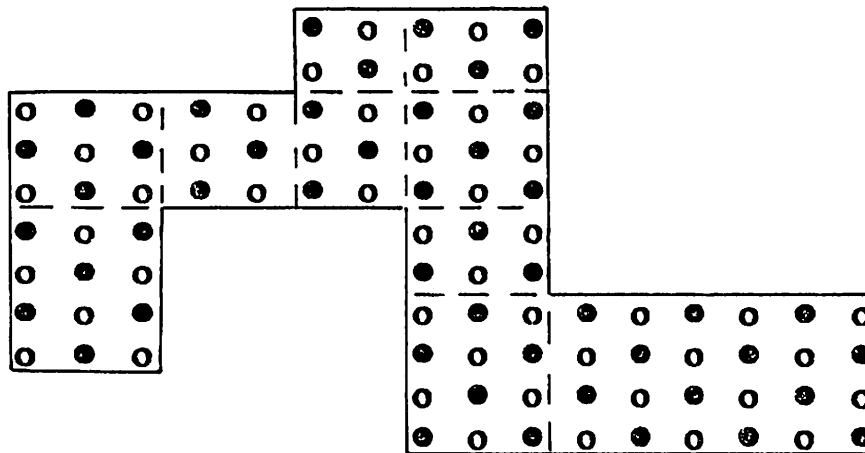


Figure 5.16. Concave-vertex decomposition of a polygon.

ure 5.16 shows a concave-vertex decomposition of size 2.

5.2.1. Evenly Decomposable Grid Graphs

Given a concave-vertex decomposable grid graph, another restriction is that all of the rectangles formed by the decomposition are n by m where n and m are even. This restriction removes the possibility of finding the type of problem shown in Figure 5.13. Also, the stripped problem will have no extra edges so it should be straightforward to solve. In order to ensure that edges $\{p,q\}$ always exist a further restriction will be that the concave-vertex decomposition is of size 4. This case is considered in Chapter Six.

5.2.2. Grid Graphs with Matchable Decompositions

In general the decomposition produces rectangles with an odd number of vertices as well as with an even number of vertices. Consider grid graphs in which each rectangle with an odd number of vertices is adjacent to another with an odd number of vertices. If these two are combined into one rectangle then the new rectangle will have an even number of vertices. Evenly decomposable grid graphs satisfy this condition trivially. Chapter Seven considers the case when the graph is restricted to those in which all of the rectangles with an odd number of vertices can be paired up. The case in which all but one odd rectangle cannot be paired up is also handled.

5.2.3. Pyramid-Shaped Grid Graphs

A *pyramid shaped grid graph* is a grid graph at least as wide at a given height as at any height above, see Figure 5.17. Pyramids have two interesting properties which simplifies the problem. The first is that because there are no relatively skinny portions, if the decomposition is of size six, then there are no problems of the type shown in Figure 5.15(c). The second is that they have the property introduced above; the odd rectangles can be paired up with at most one left over. Pyramid grid graphs are discussed in Chapter Seven.

5.2.4. L-Shaped Grid Graphs

An *L-shaped grid graph* is simply shaped like an L. There are a constant number of possible rectangle paths from a concave-vertex decomposition so all of them can be examined if necessary. Because of the restricted structure,

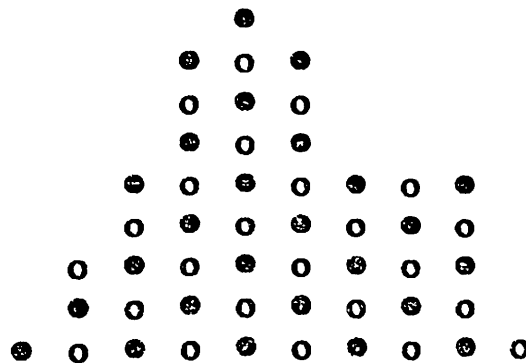


Figure 5.17. A pyramid grid graph.

there are only a few types of problems of the nature shown in Figure 5.14. The severe restriction on structure permits the problem to be solved for all sizes. The solution is found in Chapter Eight.

5.3. Summary

The diagram in Figure 5.18 shows the relationships between all the subclasses of grid graphs discussed in the thesis. The shaded area shows those classes of grid graphs for which the Hamiltonian path problem has been shown or is shown in this thesis to be polynomial. The crosshatched area shows the class which is NP-complete and the white area shows the classes for which the complexity of the problem remains unknown.

Grid Graphs

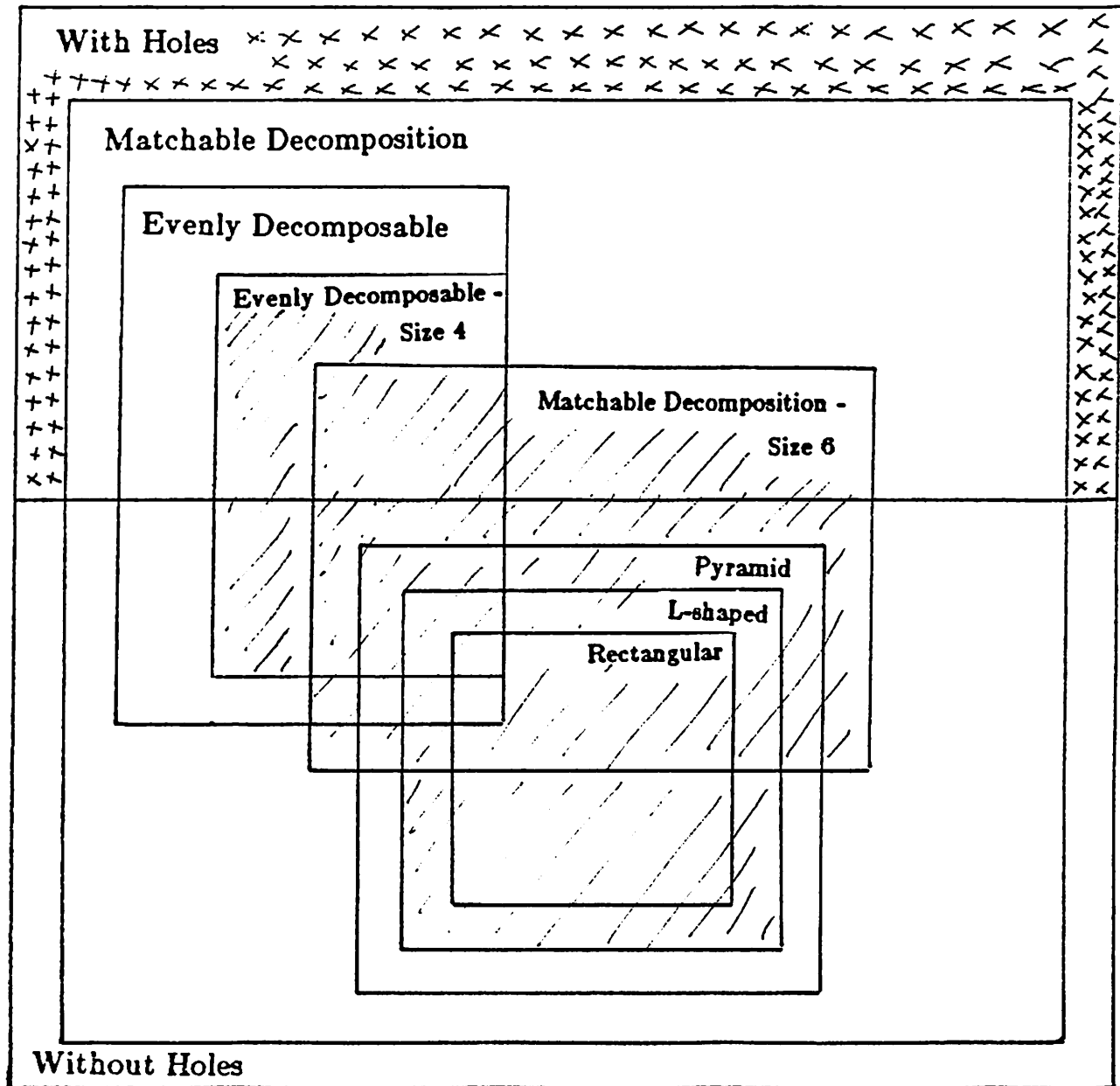


Figure 5.18. The classes of grid graphs.

CHAPTER 6

Evenly Decomposable Grid Graphs

This chapter presents the proof that it can be determined in linear time whether or not there is a Hamiltonian path in an evenly decomposable grid graph of size four with or without holes and that a path can be found if one exists. An algorithm for finding a path, based on the solution strategy of the previous chapter, is presented. Before the main result is given some terms introduced in Chapter Five are formally defined and some preliminary lemmas shown.

6.1. Preliminaries

Some terminology introduced in the previous discussions of the work on rectangular grid graphs [Itai et al.] can be generalized to apply to general grid graphs. (G,s,t) is the notation for the problem of finding a Hamiltonian path from s to t in grid graph G . (G,s,t) is said to be solvable if there is a Hamiltonian path from s to t in G . It is always assumed that G is connected. (G,s,t) is called connected if $G - \{s\}$, $G - \{t\}$, and $G - \{s,t\}$ are all connected. The Itai et al. definition of color compatibility stated in terms of rectangular grid graphs applies equally well to general grid graphs. The definition is given in Section 4.8.

The following six lemmas prove some properties of general grid graphs.

Lemma 6.1: Let G be a connected grid graph. (G,s,t) must be color compatible and connected to be solvable.

Proof: Because $G = (V,E)$ is bipartite, $V = \{V_1 \cup V_2\}$, any path alternates between members of V_1 and V_2 . If (G,s,t) is not color compatible then no alternation is possible that uses all the vertices. If (G,s,t) is disconnected then a path can traverse only one of the connected components and can never get to another. $\square$

Lemma 6.2: Let G be a connected grid graph. G can be completely specified by a rectilinear region, $P(G)$.

Proof: A grid graph is completely specified by its vertices. Let ξ be a real number, $0 \leq \xi \leq 1$. $P(G)$ is constructed by drawing lines parallel to the edges on the boundary of G but distance ξ outside G . If G contains holes then rectilinear holes are added to $P(G)$ by drawing lines parallel to the edges forming the holes in G but distance ξ inside G . Thus, $P(G)$ contains all the vertices of G and no others. $\square$

A *concave-vertex decomposition* of $P(G)$ is a partition into k rectangles, $R = \{R_1, R_2, \dots, R_k\}$, formed by constructing horizontal and vertical chords at each concave vertex on the boundary of $P(G)$ and at each convex vertex of the holes. The resulting intersection points of chords with the sides of the region and with other chords define the rectangles. The *size* of a concave-vertex decomposition is the smallest dimension of the resulting

rectangles. A grid graph with a concave-vertex decomposition of size x is denoted G^x . Figure 6.1 shows a concave-vertex decomposition of size 2.

The next lemma establishes the complexity of constructing a concave-vertex decomposition of $P(G)$. Assume that the grid graph is represented by the coordinates of its vertex set.

Lemma 6.3: The size of the concave-vertex decomposition of $P(G)$ can be determined in time linear in the number of vertices in G .

Proof: Let n be the number of vertices in G . Recall that the vertices of a grid graph lie on integer coordinates. A grid graph can therefore be represented by a matrix in which the rows of the matrix correspond to the y -axis and the columns correspond to the x -axis. The size of the matrix is determined by the largest x and y coordinates of the vertex set. The size of the matrix is thus easily determined in $O(n)$ time. Each vertex of the grid graph is placed into the matrix cell corresponding to the coordinates of the

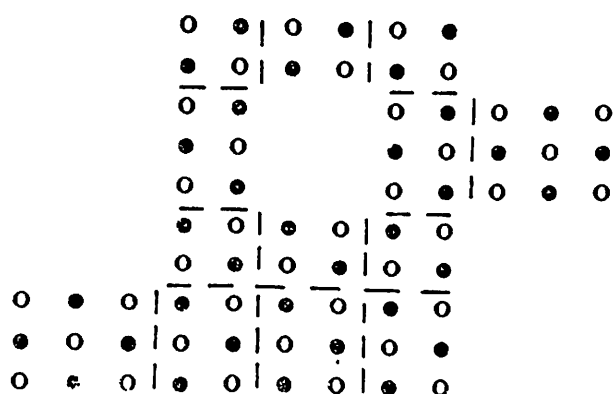


Figure 6.1. Concave-vertex decomposition of size 2.

vertex. Note that there is at most one vertex in each cell. Since each vertex can be placed in constant time and the entire matrix need not be initialized [Exercise 2.12, p. 71, Aho et al.] this can be done in $O(n)$ time. Note that the rectilinear region can be found by identifying vertices whose neighboring cells are empty.

It is now necessary to find all of the concave vertices of the boundary polygon and the convex vertices of the holes in the rectilinear region. Note that each concave and convex vertex of the rectilinear region can be represented by the vertex of the grid graph which is diagonally adjacent. Identifying such a vertex in the grid graph can be done easily by examining the eight entries in the matrix surrounding the vertex. To examine all of the vertices use depth first search. Since a grid graph has $O(n)$ edges this can be done in $O(n)$ time.

Each time a concave vertex of the boundary polygon or a convex vertex of a hole is found it is necessary to draw the chords of the decomposition. For each concave vertex it is necessary to draw two rectilinear chords separating two rows or columns of vertices. For each chord, this is done by examining the appropriate elements in the matrix proceeding up, down, right or left until an element is found which doesn't contain a vertex. When this happens either the boundary of the region or a hole has been reached. Since a vertex is adjacent to at most four sides of the decomposition and each chord may be drawn twice, once from each end, a vertex is examined at most eight times.

Thus $O(n)$ time is required to draw all of the chords of the decomposition. In the worst case each rectangle formed from the decomposition contains one vertex of G so there must be n or fewer rectangles. The size of the decomposition can then be determined by examining the dimensions of each rectangle formed and choosing the smallest. $\square$

Let G be a grid graph with n vertices and $R = \{R_1, R_2, \dots, R_k\}$ the rectangles formed by the concave-vertex decomposition. Define the *decomposition graph* of G , $D(G) = (V, E)$ where $V = \{v_1, v_2, \dots, v_k\}$ and $\{v_i, v_j\} \in E$ if R_i is adjacent to R_j in the concave-vertex decomposition of $P(G)$, $1 \leq i, j \leq k$, $i \neq j$. Two rectangles are adjacent if they share a side. Figure 6.2 shows the decomposition graph of the grid graph in Figure 6.1.

Let s and t be two vertices in G , $s \in R_\sigma$ and $t \in R_\tau$, $1 \leq \sigma \leq k$, $1 \leq \tau \leq k$. A connected path of rectangles, R_p , from R_σ to R_τ , is a sequence, $R_{p_1}, R_{p_2}, \dots, R_{p_m}$, such that $R_{p_i} \in R$, $1 \leq i \leq m$, R_{p_1} is R_σ , R_{p_j} is adjacent to $R_{p_{j+1}}$ for $1 \leq j < m$, and R_{p_m} is R_τ .

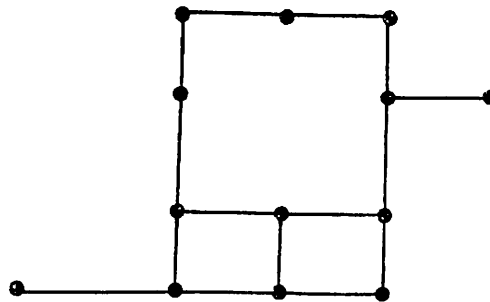


Figure 6.2. A decomposition graph.

Lemma 6.4: There is a connected path of rectangles, R_p , from R_σ to R_τ in G which can be found in $O(n)$ time.

Proof: The proof is by construction. Form $D(G)$, the decomposition graph of G . Note that $D(G)$ is connected. Let v_σ be the vertex in $D(G)$ corresponding to R_σ and v_τ be the vertex corresponding to R_τ . Since the graph is connected there is a path from v_σ to v_τ and thus a path of rectangles in G .

R_p can be found by depth first search on $D(G)$ in $O(k + e)$ time where $k = |V|$ and $e = |E|$ in $D(G)$. k is at most n and e at most $4n$ so the path can be found in time linear in the number of vertices in G . $\square$

The following lemma generalizes the principle of stripping for grid graphs whose concave-vertex decomposition is of size four.

Lemma 6.5: Let G^4 be a grid graph whose concave-vertex decomposition, R , is of size four. Let R_p be a connected path of rectangles from R_σ to R_τ in R . Let C be a connected component of $G^4 - R_p$. If (R_p, s, t) is solvable then there exists edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is an edge on the solution of (R_p, s, t) , and p' and q' are adjacent to each other in C .

Proof: Let P be the Hamiltonian path of R_p . There must be edges in P that connect the vertices on the boundary of R_p . Each component has at least one side adjacent to the boundary of R_p and this side has at least 4 vertices. There is at least one edge that can be chosen as $\{p, q\}$. $\square$

Figure 6.3 shows the worst case when s and t are on the boundary of R_p and adjacent to C .

The next lemma shows sufficient and necessary conditions for (G^4, s, t) to be connected.

Lemma 6.6: (G^4, s, t) is connected if and only if s and t are not diagonally across from one another at a convex vertex as shown in Figure 6.4.

Proof: By definition, (G^4, s, t) is connected if G^4 , $G^4 - \{s\}$, $G^4 - \{t\}$, and $G^4 - \{s, t\}$ are connected. It was assumed that G^4 is connected. Since each rectangle in the decomposition is connected to another by at least four edges, $\{s\}$, $\{t\}$, or $\{s, t\}$ can only disconnect the graph if they disconnect a rectangle.

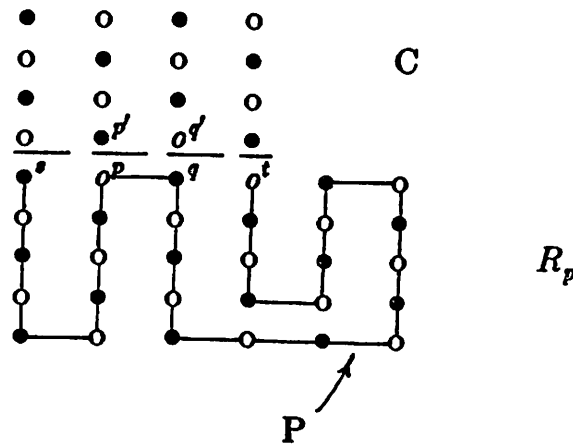


Figure 6.3. Stripping a component.

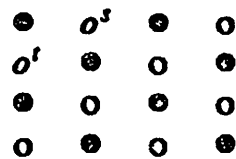


Figure 6.4. A disconnected grid graph.

A four by four or larger rectangle can only be disconnected as shown in Figure 6.4. $\square$

Note that s and t must have the same color to cause G to be disconnected. Also note that connectivity can be determined in linear time.

6.2. The Algorithm

A grid graph is said to be *evenly decomposable* if it is concave-vertex decomposable and each R_i has an even number of vertices. Let E be an evenly decomposable grid graph. (E,s,t) is called *acceptable* if it is color compatible and connected. The following lemma applies to evenly decomposable grid graphs (with or without holes) whose concave-vertex decomposition is size four.

Lemma 6.7: If (E^4,s,t) is acceptable and R_p is a rectangle path of E^4 then (R_p,s,t) is solvable.

Proof: Suppose the path R_p is given by $R_{p_1}, R_{p_2}, \dots, R_{p_m}$. For each R_{p_i} , $1 \leq i < m$, choose an edge, $\{p_i, q_i\}$ such that the following conditions hold, 1) $p_i \in R_{p_i}$ and $q_i \in R_{p_{i+1}}$, 2) neither p_i nor q_i coincides with s or t and 3) all (R_{p_i}, q_{i-1}, p_i) and (R_{p_m}, s, p_1) are color compatible. See Figure 6.5.

Because each rectangle is adjacent to the next in at least four vertices, there are four choices of $\{p_i, q_i\}$ which satisfy condition 1. Of these, condition 2 precludes at most two of them. Since each R_{p_i} is even, at least two of

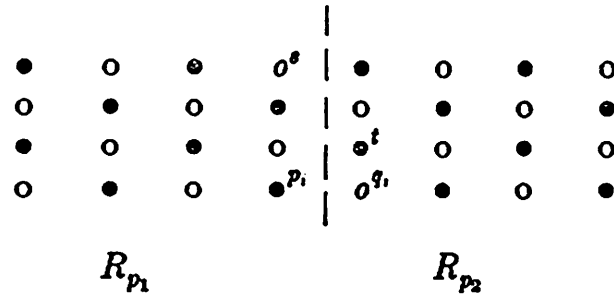


Figure 6.5. Solving a rectangle path.

the remainder must satisfy condition 3.

Solve (R_{σ}, s, p_1) , $(R_{p_i}, q_i - 1, p_i)$ for all $1 < i < m$ and (R_{τ}, q_i, t) using the algorithm of Itai et al. This is possible because condition 3 ensures that each rectangle problem is color compatible, and because there are an even number of vertices, s and t are different colors and so by Lemma 6.6 each rectangle problem is connected. $\square$

The following algorithm solves (E^4, s, t) if (E^4, s, t) is acceptable.

Algorithm 1:

Step 1. Construct a concave-vertex decomposition of E^4 .

Step 2. Find a path of rectangles, R_p , from s to t .

Step 3. Solve (R_p, s, t) .

Step 4. If $E^4 = R_p$ then (E^4, s, t) has been solved. Otherwise, determine the connected components, $C_1, C_2, \dots, C_l$, of $E^4 - R_p$.

For each C_i , $1 \leq i \leq l$, choose edges $\{p_i, p_i'\}$ and $\{q_i, q_i'\}$ such that $\{p_i, q_i\}$ is an edge on the solution of (R_p, s, t) , and p_i' and q_i' are in C_i .

Step 5. Solve all (C_i, p_i', q_i') , $1 \leq i \leq l$, using this algorithm.

6.3. Proof of Correctness

E_4 is concave-vertex decomposable by definition. Lemma 6.4 shows that there is an R_p in E . Consider Step 3. (R_p, s, t) must be color compatible because it is composed of rectangles each containing an even number of vertices. By Lemma 6.6 (R_p, s, t) must be connected because s and t are colored differently. Thus (R_p, s, t) is acceptable and by Lemma 6.7, solvable.

In Step 4 the connected components can be found by depth first search. Lemma 6.5 shows that $\{p_i, q_i\}$ can be found for any C_i . Note that because a distinct $\{p_i, q_i\}$ can be found for all C_i , a distinct side can be found for each component since each side of R_p is adjacent to only one component.

All the C_i must be color compatible because they are all even and all $\{p_i', q_i'\}$ are adjacent to each other so they must be of different colors. For this reason all C_i are also connected and thus acceptable.

Since R_p must contain at least one rectangle, each time the algorithm is called in Step 5, E^4 is smaller and eventually $R_p = E^4$ and the algorithm stops. $\square$

6.4. Timing Analysis

Steps 1 and 2 have been shown linear in Lemmas 3 and 4. Finding a p and q in Steps 3 and 4 takes constant time; choose $\{p,q\}$ at random, if p or q coincides with s or t then make a second choice and a third if necessary. No more than three choices will be required. At most $4k$ p 's and q 's need to be found ($k \leq n/16$). Determining connected components can be done in linear time using depth first search. There are k rectangle problems to be solved that can be done in linear time.

Since acceptability can be determined in linear time we have:

Theorem 1: Whether or not there is a Hamiltonian path in a grid graph with or without holes that has an even decomposition of size four can be determined and a path can be found if one exists in time linear in the number of vertices.

CHAPTER 7

Pyramid Grid Graphs

In this chapter it is shown how to determine in linear time whether or not there is a Hamiltonian path in a concave-vertex decomposable grid graph of size six without holes that are pyramid shaped and to find a path if one exists. If the decomposition produces only rectangles with an even number of vertices then the problem can be solved by Algorithm 1. In general, however, rectangles with an odd number of vertices will be formed as well. Chapter Five showed some examples where rectangles with an odd number of vertices are troublesome.

Observe that two rectangles with an odd number of vertices side by side make up one rectangle with an even number of vertices. It is shown that if all of the rectangles can be combined into rectangles with an even number of vertices then Algorithm 1 can be used to find a solution. A modified Algorithm 1 is shown to find a solution when all but one of the rectangles can be combined into rectangles with an even number of vertices. Finally, it will be shown that all, or all but one, of the rectangles in the decomposition of pyramid grid graphs can be combined into rectangles with an even number of vertices.

7.1. Bipartite Matchings

Let $G = (V_1 \cup V_2, E)$, where V_1 are vertices colored white and V_2 are vertices colored black, be a concave-vertex decomposable grid graph with decomposition, $R = \{R_1, R_2, \dots, R_k\}$, and the decomposition graph, $D(G)$. Color the vertices of $D(G)$, $v_1, v_2, \dots, v_k$ in the following manner: v_i is black (denoted, $\bullet$) if R_i has a majority of black vertices and white (denoted, $\circ$) if R_i has a majority of white vertices, $1 \leq i \leq k$. Vertices that are not colored are called neutral (denoted n). Note that $D(G)$ has no edge connecting vertices of the same color since a rectangle with a majority of black vertices must be adjacent to either a neutral rectangle or to a rectangle with a majority of white vertices.

A *nearest orthogonal neighbor* of a vertex, v_i , in $D(G)$ is the nearest black or white vertex, v_j , such that there is a connected path from v_i to v_j which goes in only one direction, straight up, down, left or right. Note that v has 0, 1, 2, 3 or 4 nearest orthogonal neighbors.

A *grouping graph*, $M(G)$, is constructed from $D(G)$ where $M(G)$ contains the non-neutral vertices of $D(G)$ and there is an edge between two vertices in $M(G)$ if they are nearest orthogonal neighbors in $D(G)$. Figure 7.1 shows an example of G , $D(G)$ and $M(G)$. Note that $M(G)$ is bipartite.

Lemma 7.1: $M(G)$ is a planar graph.

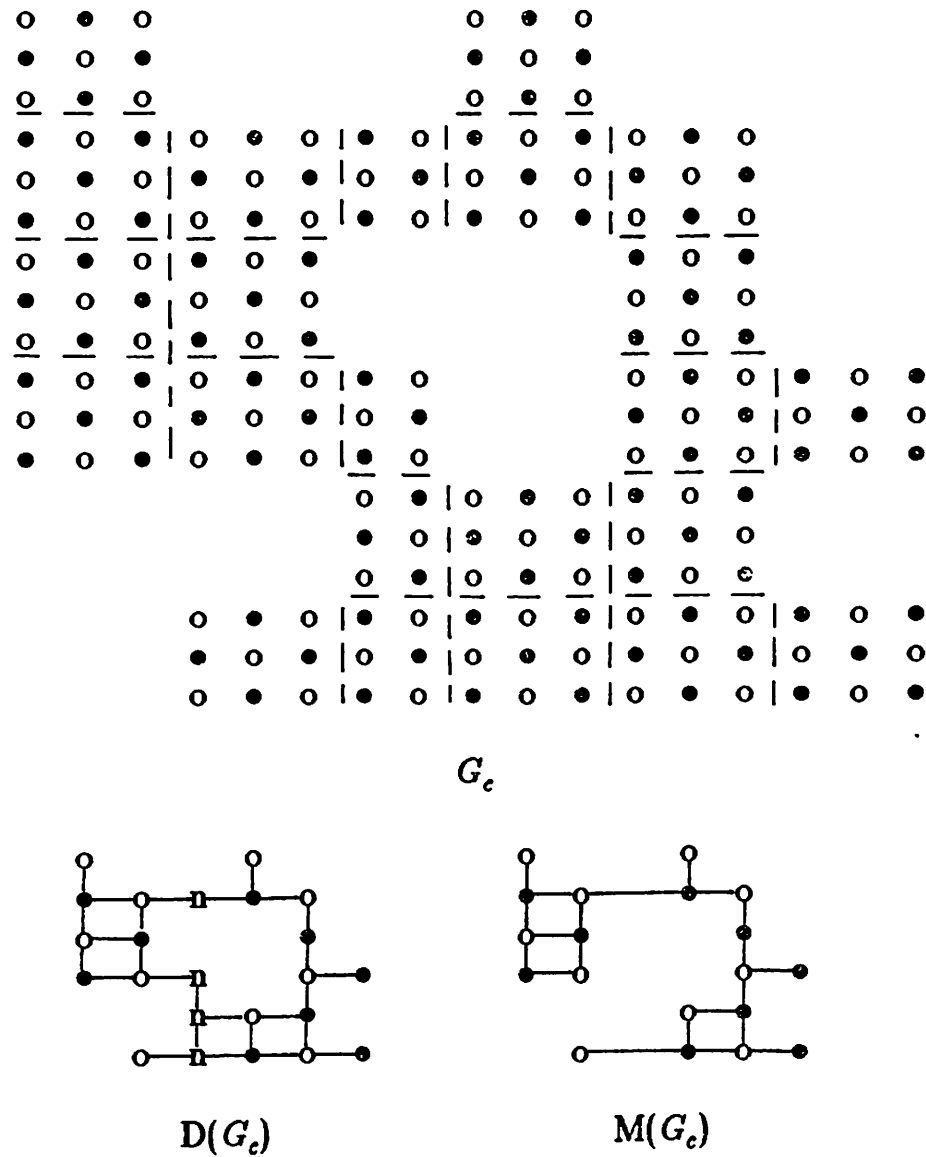


Figure 7.1. A grouping graph.

Proof: It is necessary to show that $M(G)$ can be constructed so as to contain no crossing edges. Note that the only way that edges can cross is if two sets of nearest orthogonal neighbors are connected by paths going through a neutral vertex. However, this is not possible since all vertices in the same connected column or row must be neutral since the neutral rectangle

cannot be odd by odd. Thus $M(G)$ can contain no crossing edges. $\square$

A *perfect matching* is a subset of the edges of a graph such that each vertex is incident to one and only one edge. The matching is not perfect if there is some vertex that is not matched. A matching with one vertex unmatched is called a *near perfect matching*. Figure 7.2 shows a near perfect matching of the graph $M(G)$ from Figure 7.1. Note that there is no perfect matching in $M(G)$.

Two vertices, v_i and v_j , of $M(G)$ which are incident to the same edge in a matching of $M(G)$ are called a *matched pair*. Note that each matched pair corresponds to two rectangles, R_i and R_j , in the decomposition of G . R_i and R_j are either adjacent (as in pair 1 in Figure 7.2) or they are at either end of a set of adjacent neutral rectangles (as in pair 9 in Figure 7.2). The rectangles in G corresponding to v_i and v_j plus any neutral rectangles in between is denoted $R_{i,j}$. $R_{i,j}$ is said to be adjacent to a rectangle in G if any member of

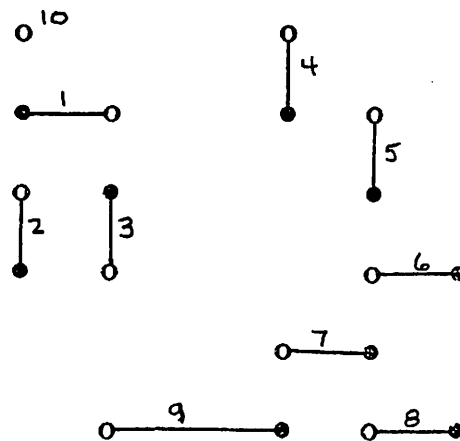


Figure 7.2. A near perfect matching.

$R_{i,j}$ is adjacent to it.

An *adjacency graph*, $A(G)$, can be constructed for each perfect or near perfect matching of $M(G)$. $A(G)$ has one vertex for each matched pair in the matching of $M(G)$, one vertex for each unmatched vertex in the matching of $M(G)$, and one vertex for each neutral vertex in $D(G)$ which is not part of any $R_{i,j}$. The edges of $A(G)$ join vertices if their corresponding rectangles or corresponding rectangle sets, $R_{i,j}$, in G are adjacent. Note that $A(G)$ is connected. Figure 7.3(a) shows $A(G)$ for the matching in Figure 7.2.

A new decomposition of G can be defined using $A(G)$; each vertex in $A(G)$ corresponds to a set of one or more rectangles in the decomposition of G . Call the new decomposition $N(G)$. Figure 7.3(b) shows $N(G)$ obtained from $A(G)$ in Figure 7.3(a).

Note that the union of the rectangles in the rectangle set, $R_{i,j}$, of each matched pair has an even number of vertices. Thus if $M(G)$ has a perfect matching, each rectangular grid graph in G corresponding to a rectangle in $N(G)$ has an even number of vertices. Using this new decomposition results in a problem similar to that of evenly decomposable grid graphs of size four for which there is an algorithm, Algorithm 1.

If $M(G_c)$ has a near perfect matching then there is one rectangular grid graph, R_i , in G corresponding to a rectangle in $N(G)$ which has an odd number of vertices. R_i is the rectangle corresponding to the unmatched vertex, v_i . Consider a path of rectangles corresponding to vertices in $A(G)$ from

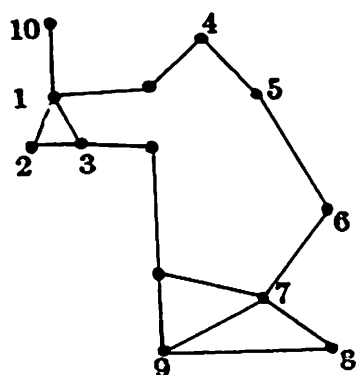
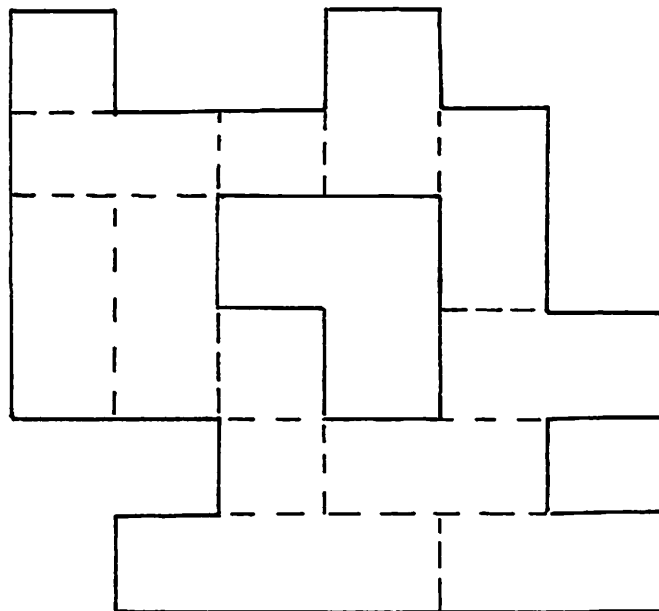
(a). $A(G)$ (b). $N(G)$

Figure 7.3. An adjacency graph and the new decomposition.

the vertex corresponding to the rectangle containing s to the vertex corresponding to the rectangle containing t . Suppose R_i is not on the path of rectangles. Algorithm 1 would treat such a rectangle as a piece of a connected component. Recall that connected components are treated in a similar process to stripping. However, R_i cannot be stripped because it has an odd number of vertices. Suppose R_i is on the path of rectangles. In this case it can be treated as in Algorithm 1. Thus the approach used is to force v_i to be part of the rectangle path.

Let $RN = \{RN_1, RN_2, \dots, RN_m\}$, be the set of rectangles in $N(G)$. RN_σ is the rectangle corresponding to the rectangular grid graph containing s and

RN_τ is the rectangle corresponding to the rectangular grid graph containing t . Let v_s be the vertex in $A(G)$ corresponding to RN_σ and v_t the vertex corresponding to RN_τ .

Lemma 7.2: Given G , if $M(G)$ has a perfect matching then there is a path of connected rectangles, RN_w , from RN_σ to RN_τ . If $M(G)$ has a near perfect matching with v_i the unmatched vertex, then there is a walk (not necessarily vertex disjoint path) of connected rectangles, RN_w , from RN_σ to RN_τ that goes through RN_i .

Proof: Construct $A(G)$, the adjacency graph of G , for any perfect or near perfect matching in $M(G)$. Note that since $A(G)$ is connected there is a path between any two vertices. If there is a perfect matching in $M(G)$ find the path between v_s and v_t . The corresponding rectangles form RN_w . If there is not a perfect matching then find the path from v_s to the unmatched vertex, v_i , and from v_i to some vertex, v_j such that v_j is the only vertex on the path from v_s to v_i . The paths from v_s to v_i , from v_i to v_j and from v_j to v_t form a walk and the corresponding rectangles form RN_w . $\square$

The following lemma shows the relationship between the solvability of (RN_w, s, t) and perfect and near perfect matchings. Consider the decomposition graph in Figure 7.4(a) with the matching indicated. Figure 7.4(b) shows the grid graph corresponding to the decomposition graph. Suppose s and t are as shown. Note that this problem has no solution. It turns out that if G

is restricted to be size six then problems of this nature do not exist. Let G^6 be a grid graph with a concave-vertex decomposition of size six. G^6 is called *acceptable* if it is color compatible and connected. Let RN_w be the walk constructed as in Lemma 7.2.

Lemma 7.3: If $M(G^6)$ has a perfect or near perfect matching and (G^6, s, t) is acceptable then (RN_w, s, t) is solvable.

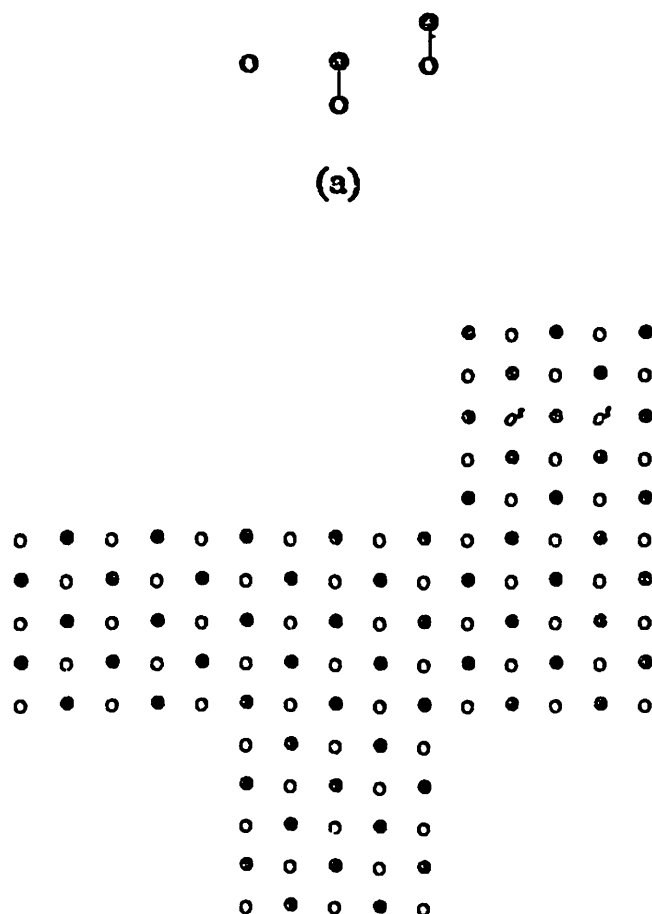


Figure 7.4. An unsolvable problem.

Proof: The proof is constructive. There are two cases.

Case 1: Suppose s and t are of different colors. Because (G^6, s, t) is color compatible, there must be an equal number of black and white vertices in $D(G^6)$. Since it is assumed that there is either a perfect or near perfect matching, there must be a perfect matching and each rectangular grid graph corresponding to a rectangle in $N(G^6)$ has an even number of vertices.

If s and t are in the same rectangle then RN_w consists of a single rectangle. (RN_w, s, t) is solvable because it must be color compatible and, since s and t are different colors, connected.

If s and t are not in the same rectangle, then for each RN_{w_i} , $1 \leq i < m$, choose an edge $\{p_i, q_i\}$ such that, 1) $p_i \in RN_{w_i}$ and $q_i \in RN_{w_{i+1}}$, 2) no p_i or q_i coincides with s or t and 3) all (RN_{w_i}, q_{i-1}, p_i) and (RN_{w_i}, s, p_1) are color compatible, see Figure 7.5. Because each rectangle is adjacent to the next in at least six vertices there are six choices of $\{p_i, q_i\}$ which satisfy condition 1. Of

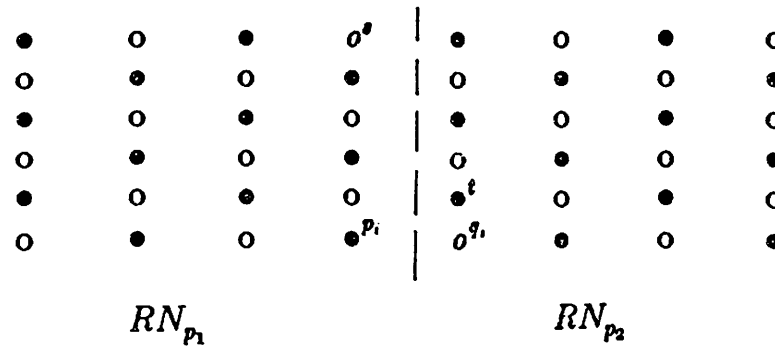


Figure 7.5. Choosing p and q .

these, condition 2 precludes at most 2 of them. Since each RN_{w_i} is even, at least three of the remainder must satisfy condition 3. Solve (RN_{σ}, s, p_1) , $(RN_{w_i}, q_i - 1, p_i)$ for all $1 < i < m$ and $(RN_{\tau}, q_{i,t})$ using the algorithm of Itai et al. This is possible because each rectangle must be color compatible by constructions and connected since s and t are different colors.

Case 2: Suppose s and t are the same color (assume black). Because (G^6, s, t) is color compatible, there must be one more rectangle with a majority of black vertices than there are with a majority of white. Since it is assumed that there is either a perfect or near perfect matching in $M(G^6)$, there must be a near perfect matching. Let v_i be the unmatched vertex and RN_i the corresponding rectangle. Note that RN_i has a majority of black vertices. There are three sub-cases.

a) s and t are in RN_i . RN_w is a single rectangle; $RN_w = RN_i$. Since RN_w has a majority of black vertices and s and t are black, (RN_w, s, t) must be color compatible. (RN_w, s, t) must be connected because in order for s and t to disconnect a corner vertex they must be the opposite color of the corner vertex. This cannot occur when the rectangle has an odd number of vertices and the problem is color compatible. Thus (RN_w, s, t) is solvable.

b) s and t are in different rectangles. If the walk is vertex disjoint then solve as in case 1. Note that the choice of p and q in RN_i will be vertices with the same color. See Figure 7.6.

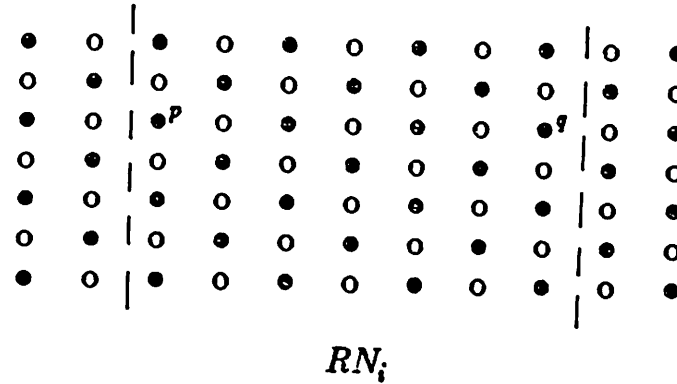


Figure 7.6. Choosing p and q in the non-grouped rectangle.

If the walk is not vertex disjoint, note that the walk doubles back on itself as shown in Figure 7.7. Solve v_s to v_{j-1} and v_i to v_{j+1} as in Case 1. It remains to solve v_j to v_i . This segment of the path is composed of rectangular problems of the following form: the path enters RN_k at one side, say side x , leaves through a side other than x , say side y , returns through side y and exits through any side, say z , where RN_k is the rectangle corresponding to v_k , where v_k is any vertex between v_i and v_j inclusive. Side y is chosen based on adjacency to the next rectangle in the path. Sides x and z can be any of the other three sides. Figure 7.8(a) shows one example when RN_k is a matched

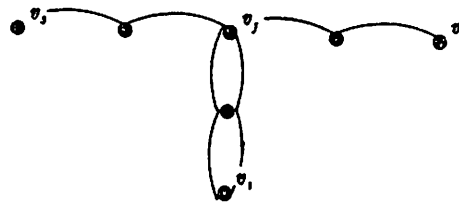


Figure 7.7. A walk in a graph with a near matching.

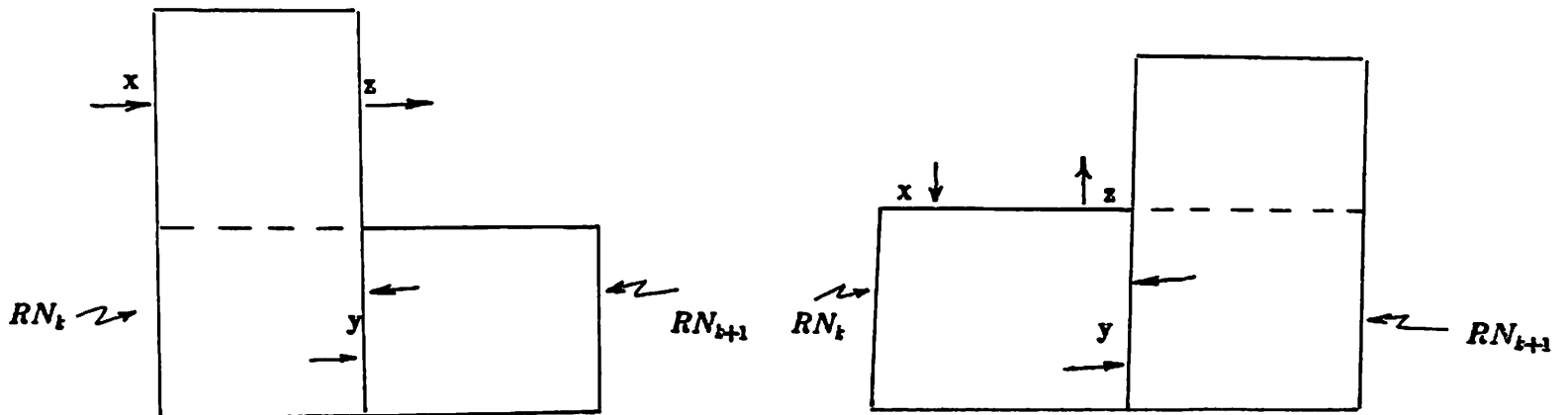
set and one example where RN_k is a neutral rectangle. These two examples differ only in the size of RN_k so consider the smallest RN_k which is six by six when RN_k is a neutral rectangle. Note that leaving and returning from side y is equivalent to adding an extra edge from the exiting point to the returning point and forcing any path to use that edge. Also note that entering side x is equivalent to starting a path at some vertex on side x and that leaving through side z is equivalent to finishing a path at some vertex on side z . Thus, each RN_k can be reformulated into the following problem, an example of which is shown in Figure 7.8(b):

Given (R,s,t) , a six by six or larger rectangular grid graph with an even number of vertices and s and t the same color, does there exist a set E' containing a single edge on any side of the boundary of R such that (R_e,s,t) is solvable where R_e is $R \cup E'$ and (R_e,s,t) is called solvable if there is a Hamiltonian path from s to t that goes through all edges in E' .

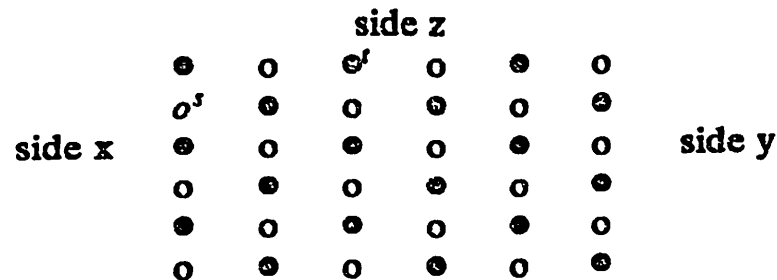
A proof that such a set exists can be found in Appendix A, Lemma A.1.

c) s and t are in the same rectangle, RN_j $j \neq i$. RN_w consists of a walk from v_j to v_i and back to v_j where v_j contains s and t . First suppose that (RN_j,s,t) is connected. Then solve as in Case 2(b) for v_j to v_i .

If (RN_j,s,t) is disconnected then since G^6 is acceptable, the disconnected vertex must be adjacent to another rectangle, say RN_z . If RN_z is on the rectangle path then choose E' to be the edge $\{p,q\}$ as in Figure 7.9. A Hamiltonian path which goes through E' can be found, see Lemma A.2 in Appendix A. The remainder of the path is solved as in Case 2(b) for v_j to v_i .



(a)



(b)

Figure 7.8. Extra edge problems.

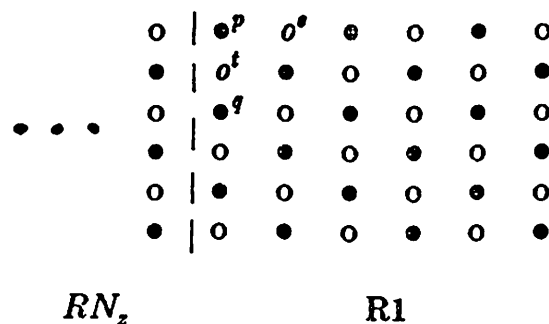


Figure 7.9. Disconnected walk.

If RN_z is not on the path then add it to the path and solve it first by using the edge, $\{p,q\}$, shown in Figure 7.10(a). $\{p',q'\}$ is the edge adjacent to $\{p,q\}$

in RN_z . RN_z has an even number of vertices since it cannot be RN_i because it is not on the path. Thus, (RN_z, p', q') is color compatible and connected and therefore solvable. To solve (RN_j, s, t) with the extra edge we have to solve problems as shown in Figure 7.10(b), (c) and (d). Since all of the problems can be solved (see Lemma A.3 in Appendix A) it remains to solve problems of the form in Case 2(b) for v_{j+1} to v_i . $\square$

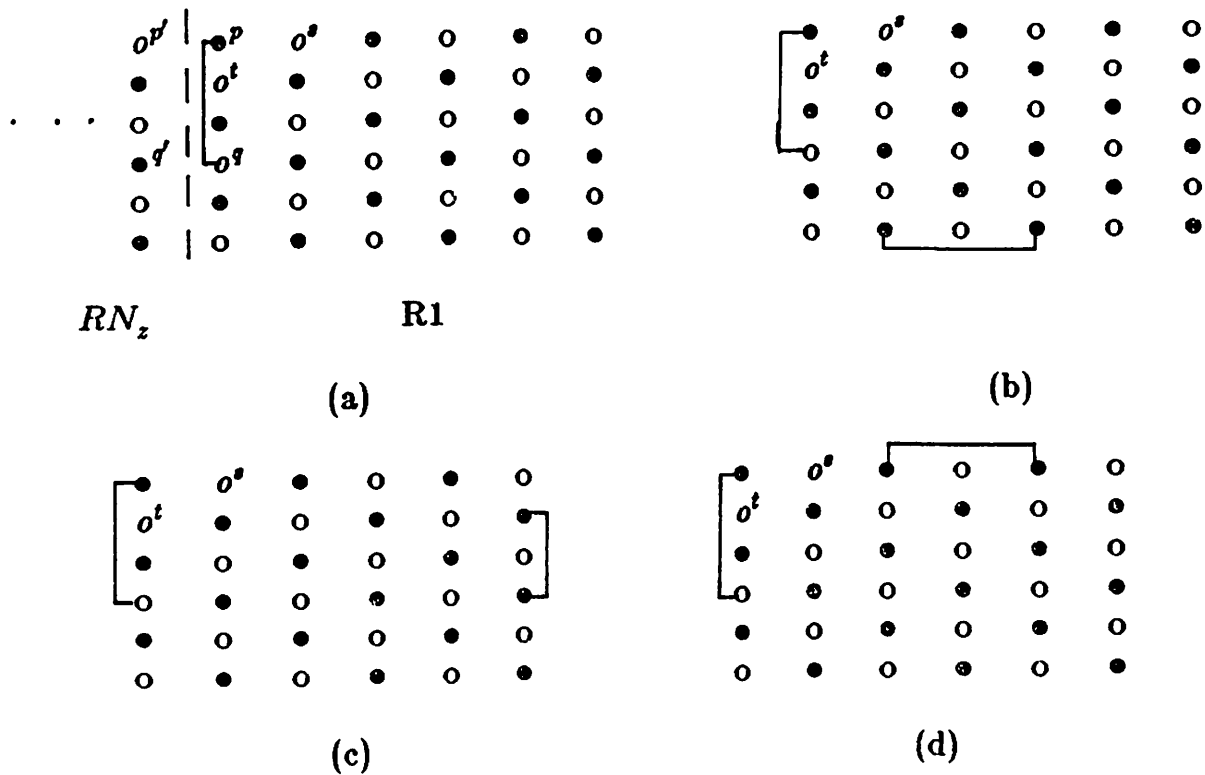


Figure 7.10. Extra edge problems.

7.1.1. The Algorithm

The following algorithm solves (G^6, s, t) if $M(G^6)$ has a near or perfect matching and (G^6, s, t) is acceptable.

Algorithm 2:

Step 1. Construct a concave-vertex decomposition of G^6 .

Step 2. Construct $M(G^6)$, the grouping graph of G^6 , and find the best matching in $M(G^6)$.

Step 3. Create an adjacency graph, $A(G^6)$, from the matching.

Step 4. Find a rectangle walk, RN_w , from s to t which passes through the non-grouped rectangle if there is one.

Step 5. Solve (RN_w, s, t) .

Step 6. If $G^6 = RN_w$ then (G^6, s, t) has been solved. Otherwise, determine the connected components, $C_1, C_2, \dots, C_l$, of $G^6 - RN_w$. For each C_i , $1 \leq i \leq l$, choose edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is an edge on the solution of (R_p, s, t) , and p' and q' are in C_i .

Step 7. Solve all (C_i, p', q') using this algorithm.

7.1.2. Proof of Correctness

G^6 is concave-vertex decomposable by definition so $D(G^6)$ can be constructed and from that $M(G^6)$. $M(G^6)$ is assumed to have a perfect or near perfect matching. Thus $A(G^6)$ can be constructed. Lemma 6.8 shows that $A(G^6)$ must have a rectangle walk. (RN_w, s, t) is solvable by Lemma 7.3. In Step 6 the connected components can be found by depth first search. Lemma 6.5 shows that $\{p, q\}$ can be found for any C_i . Note that a distinct side can be found for each component since each side of RN_w is adjacent to only one component so a distinct $\{p, q\}$ can be found for all C_i . All the C_i must be color compatible because they are composed of matched pairs and neutral rectangles and thus contain an even number of vertices. p and q are different colors because they are adjacent to one another. For this reason all C_i are also connected and thus acceptable. All $M(C_i)$ have a perfect matching because they are composed of matched pairs and neutral rectangles. Since RN_w must contain at least one rectangle, each time the algorithm is called, G^6 is smaller and eventually $RN_w = G^6$ and the algorithm stops. $\square$

7.1.3. Timing Analysis

Step 1 requires linear time by Lemmas 6.2 and 6.3. To construct $M(G^6)$ examine each vertex in $D(G^6)$ and determine whether it is the nearest non-neutral neighbor of another vertex. This can be done in linear time using

depth first search. The time required for the bipartite matching problem is $O(V^{1/2} + E)$ [Hopcroft and Karp]. In grid graphs $E < 4n$ so $O(n)$ is sufficient. Step 3 clearly requires linear time. There are at most n p 's and q 's and u 's and v 's to choose in Steps 4 and 5 and each one can be found in constant time. There are then k rectangles to be solved each taking linear time using the algorithm of Itai et al.

Since acceptability can be determined in linear time, the following theorem can now be stated.

Theorem 2: Whether or not there is a Hamiltonian path in a concave-vertex decomposable grid graph of size six with or without holes with a perfect or near perfect matching can be determined and a path can be found if one exists in time linear in the number of vertices.

7.2. Pyramids

A polygon is a *pyramid* if it is at least as wide at a given height as at any height above. If Y is a concave-vertex decomposable grid graph without holes whose representative region, $P(Y)$, has a pyramid as its boundary polygon then Y is called a pyramid grid graph. Note that if all the rectangles in the decomposition have an even number of vertices then Theorem 1 shows that there is an algorithm to solve the Hamiltonian path problem. However, it is necessary to consider pyramid grid graphs whose concave-vertex decompositions are made up of both odd and even rectangles. The next three

lemmas show that all pyramid grid graphs have a perfect or near perfect matching.

Lemma 7.4: The decomposition graph, $D(G)$, of a grid graph, G , has the property that the color of the nearest orthogonal neighbor of a non-neutral vertex, v_i , in any of the four directions is the opposite color of v_i if such a neighbor exists.

Proof: Suppose a vertex, v_i , is colored. It must correspond to a rectangle with an odd number of vertices. Such a rectangle has dimensions, m and n , where m and n are odd numbers. Call such a rectangle an odd by odd rectangle. Consider the rectangle below it. This rectangle is either odd by odd, or odd by even. If it is odd by odd then it must have the opposite color. If it is odd by even then it is a neutral rectangle and the same reasoning can be applied to the rectangle below that, if one exists, ie. it must be either odd by even, or odd by odd and of the opposite color. The argument for the other directions is similar. $\square$

Lemma 7.5: The grouping graph of a pyramid grid graph, $M(Y)$, is a pyramid grid graph without holes.

Proof: Clearly, $D(Y)$ is a pyramid. If all vertices in $D(Y)$ are non-neutral then $D(Y) = M(Y)$ and $M(Y)$ is thus a pyramid. Suppose $D(Y)$ contains a neutral vertex, v . If v is below a non-neutral vertex then it has dimensions (m,n) , m odd and n even. Call this an odd by even rectangle. Thus all vertices in the same row as v must be either odd by even or even by

even. In either case they are all neutral rectangles. The entire row is not in $M(Y)$ so $M(Y)$ is still a pyramid. If v is below a neutral vertex, then it is either odd by even, even by odd, or even by even. If it is odd by even then the entire row containing v consists of even rectangles and the entire row is not in $M(Y)$. If it is even by odd then the entire column must consist of even rectangles and the entire column is not in $M(Y)$. If it is even by even then the entire row and column must consist of even rectangles and neither the row nor column is in $M(Y)$. If v is not below anything it starts a neutral row or neutral column or both. Thus $M(Y)$ must be a pyramid grid graph without holes. $\square$

The following labelling scheme will be used to label the rows of vertices in $M(Y)$. Label each row by the absolute value of $|S_1| - |S_2|$ where $S_1 \subseteq V_1$ consists of all of the vertices in V_1 in the rows above and including the one to be labelled and $S_2 \subseteq V_2$ consists of all of the vertices in V_1 in the rows above and including the one to be labelled. Figure 7.11 shows a labelled grouping graph of a pyramid grid graph.

Lemma 7.6: Let l_i be the label of row i . l_i differs from l_{i+1} by at most one for each row of a pyramid grid graph without holes.

Proof: In order to change the value of a label from one row to the next it is necessary to add more of one color vertex than the other color. In one row it is possible to add at most one more of one color than the other. This occurs when both endpoints of the row are different colors. Thus the largest

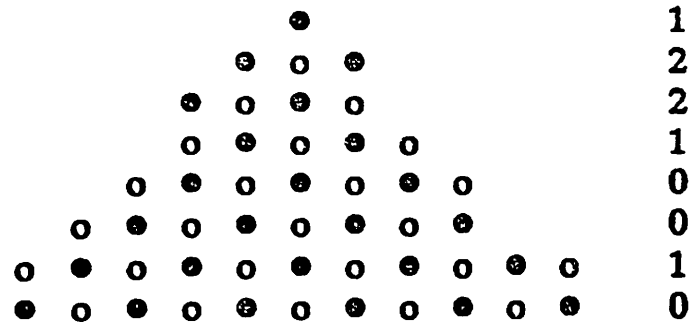


Figure 7.11. A labeled pyramid grid graph.

change is one. $\square$

Lemma 7.7: Given (Y, s, t) , a color compatible pyramid problem without holes, and $M(Y)$, the corresponding grouping graph, if s and t are of different colors then there is a perfect matching in $M(Y)$. If s and t are the same color then there is a near perfect matching in $M(Y)$.

Proof: There are two cases.

Case 1: s and t are of different colors. It is required to show that there is a perfect matching in $M(Y)$. It is shown how to match all of the vertices in any row. Let k be the number of a row of $M(Y)$ where, for the top row, $k = 1$. It is shown by induction on k , that either all of the vertices in row k have been matched with vertices in the previous row or they can be matched with other vertices in row k or with vertices in the succeeding row and that all of the vertices in row k which have been matched in the previous row are consecutive, ie. they are each separated by one vertex of the opposite color.

Base Case: $k = 1$. It is necessary to match each vertex in the first row. Note that the label of row 1 is 1 or 0 depending on whether there is an odd or even number of vertices. If the label is 0 there is an even number of vertices. Starting from the left, each vertex is matched with its neighbor to the right. If the label is 1 then there is one more vertex of one color than the other. Match neighboring vertices starting from the left. The rightmost vertex is unmatched. Match this vertex with the one below it. Lemma 7.5 shows that this vertex exists and Lemma 7.4 shows that the vertex is the opposite color.

Inductive Step: Assume that the first k rows all have their vertices matched. It is necessary to show how all of the vertices in row $k + 1$ can be matched. Note that some of the vertices in row $k + 1$ have already been matched with the vertices in row k . The number of matched vertices in row $k + 1$ equals l_k , the label of row k . Note also that the matched vertices all have the same color and assume that they are consecutive, ie. they are each separated by one vertex of the opposite color as shown in Figure 7.12.

In order to match all of the remaining unmatched vertices in row $k + 1$ choose l_{k+1} vertices to be matched with vertices in row $k + 2$. (Note that if there is no row $k + 2$ then $l_{k+1} = 0$). Note that these vertices are the opposite color of the ones already matched. Suppose $l_{k+1} > l_k$. Choose the vertices to be those on either side and between the ones already matched. Suppose $l_{k+1} < l_k$. Choose the vertices to be those between the ones already

matched. Suppose $l_{k+1} = l_k$. Choose the vertices to be those between and to one side (choose the side so that the vertex at the end of the row on that side is the same color as the vertex to be matched) of the ones already matched. Note that the matched vertices in row $k + 2$ are consecutive, ie. they are each separated by one vertex of the opposite color.

Consider the vertices which are still unmatched in row $k + 1$. The matched vertices consist of consecutive vertices and the unmatched ones must lie on either side of the matched vertices. It is now shown that these must come in pairs so they can be matched together. Suppose $l_{k+1} > l_k$. The set of matched vertices has the majority color on either end. Since the entire row has the majority color on either end, there must be an even number on either of the matched vertices. The same applies for $l_{k+1} < l_k$. If $l_{k+1} = l_k$ then the set of matched goes from black to white and the entire row goes from black to white so there must be an even number of unmatched vertices on either side of the matched vertices.

Case 2: s and t are the same color (say black). Note that there is one more black vertex than white vertex in $D(Y)$ so it is necessary to find a near perfect matching. The approach is to find a vertex which will not be matched

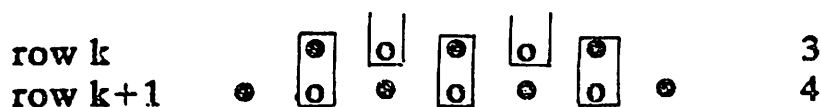


Figure 7.12. The inductive step.

and show that the remaining vertices can be matched. There must be a vertex, v_b , in $D(Y)$ that is colored black and the removal of which leaves a pyramid grid graph. If the apex of the pyramid is black then this vertex can be v_b . If the apex is white then at some place further in the pyramid there must be a row of vertices with one endpoint black and no vertex above it, see Figure 7.13. This vertex can be v_b . If v_b is removed from $D(Y)$ then $M(Y)$ (constructed from the modified $D(Y)$) has the same number of vertices of each color. The first part of the proof shows that $M(Y)$ has a perfect bipartite matching. $\square$

Let Y^6 be a concave-vertex decomposable pyramid grid graph of size six without holes. The following theorem may now be stated.

Theorem 3: Whether or not there is a Hamiltonian path in a pyramid grid graph without holes that has a concave-vertex decomposition of size six can be determined and a path can be found if one exists in time linear in the number of vertices.

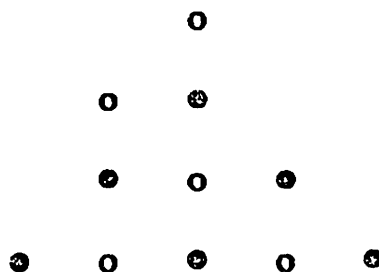


Figure 7.13. Pyramid with a near perfect matching.

CHAPTER 3

L-shaped Grid Graphs

In this chapter it is shown that whether or not there is a Hamiltonian path in an L-shaped grid graph without holes can be determined and a path can be found if one exists in time linear in the number of vertices.

An L-shaped grid graph is a grid graph shaped like an L. Clearly, the L-shaped grid graphs are a sub-class of the pyramid grid graphs. Thus, Theorem 3, which states that whether or not there is a Hamiltonian path in a pyramid grid graph without holes that has a concave-vertex decomposition of size six can be determined and a path can be found if one exists in time linear in the number of vertices, also applies to L-shaped grid graphs without holes and with a concave-vertex decomposition of size six. It remains to consider L-shaped grid graphs without holes whose concave-vertex decomposition is smaller than size six.

Algorithm 2, which solves the problem for large pyramid grid graphs, relies on the fact that any matching in the grouping graph and any walk in the corresponding adjacency graph, leads to a Hamiltonian path. To solve the problem for small L-shaped grid graphs it is shown that in the worst case it is necessary to examine all possible matchings and paths in the decomposition graph. Because there are a constant number of vertices in the decomposition

graph of an L-shaped grid graph, this can still be done in linear time.

8.1. Large L-shaped Grid Graphs

Let L^4 be an L-shaped grid graph without holes whose concave-vertex decomposition is of size four. (L^4, s, t) is called *acceptable* if it is color compatible and connected. First, it is shown that, without modification, Algorithm 2 solves (L^4, s, t) if (L^4, s, t) is acceptable.

Lemma 8.1: Whether or not there is a Hamiltonian path in an acceptable L-shaped grid graph without holes that has a concave-vertex decomposition of size four, (L^4, s, t) , can be determined a path can be found if one exists in time linear in the number of vertices.

Proof: Consider Algorithm 2 which is reproduced below.

Algorithm 2:

- Step 1. Construct a concave-vertex decomposition of G_c^6 .
- Step 2. Construct $M(G_c^6)$, the grouping graph of G_c^6 , and find the best matching in $M(G_c^6)$.
- Step 3. Create an adjacency graph, $A(G_c^6)$, from the matching.
- Step 4. Find a rectangle walk, RN_w , from s to t which passes through the non-grouped rectangle if there is one.
- Step 5. Solve (RN_w, s, t) .
- Step 6. If $G_c^6 = RN_w$ then (G_c^6, s, t) has been solved. Otherwise, determine the connected components, $C_1, C_2, \dots, C_l$, of $G_c^6 - RN_w$. For each C_i , $1 \leq i \leq l$, choose edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is an edge on the solution of (R_p, s, t) , and p' and q' are in C_i .

Step 7. Solve all (C_i, p', q') using this algorithm.

Algorithm 2 is now shown correct for (L^4, s, t) . By Lemma 6.3, Step 1 is possible for any size of grid graph. It is shown in Lemma 7.7 that the grouping graphs of pyramid grid graphs without holes whose concave-vertex decomposition is of size six have a perfect or near perfect matching. Since it is the parity of the rectangles and not the size that determines whether a matching exists, $M(L^4)$ must have a near perfect matching. Thus Step 2 can be achieved in linear time. Clearly Step 3 is also possible in linear time. By Lemma 7.2, Step 4 is possible for any size of grid graph.

Consider Step 5. Lemma 7.3 shows that this is possible for (G_c^6, s, t) . Recall the proof of Lemma 7.3. It is composed of two cases. The first case deals with perfect matchings. Except for the size, the proof is essentially the same as the proof of Lemma 6.7 of the result for evenly decomposable grid graphs of size four. Thus by Lemma 6.7, (RN_w, s, t) is solvable for size four.

The second case is separated into three sub-cases depending on the location of s and t . The first of these deals with s and t in the non-grouped rectangle, v_i . (v_i, s, t) is solvable for size four. The second sub-case deals with s and t in different rectangles. The proof refers to Lemma A.1 which can be shown for size four except in the case shown in Figure 8.1(a). In the third sub-case s and t are in the same rectangle. The proof refers to Lemmas A1.2 and A1.3 which can be shown for size four except in the case of Lemma A.3 shown in Figure 8.1(b), and the case of Lemma A.3 shown in Figure 8.1(c).

It is now shown that none of these cases can arise in L-shaped grid graphs.

Consider Figure 8.1(a). Recall from Lemma 7.3 that this case arises only when s and t are the same color. In this case s and t are black. However, the rectangle connected by the extra edge, $\{p,q\}$, is either neutral or white. If it is neutral then the third rectangle must also be neutral and if it is white the third rectangle is either black or neutral. In any case (L^4, s, t) would not be color compatible since both s and t black imply it must be black. But the lemma assumes (L^4, s, t) is color compatible. Consider Figure 8.1(b). This can never arise because the decomposition of an L-shaped grid graph never has three rectangles in a row. Consider Figure 8.1(c) in which u and v are both white but in which there is only one free white vertex on the side of the rectangle. This case never arises because the rectangle connected by edge $\{u, v\}$ must be odd by even, or even by even so it must be neutral.

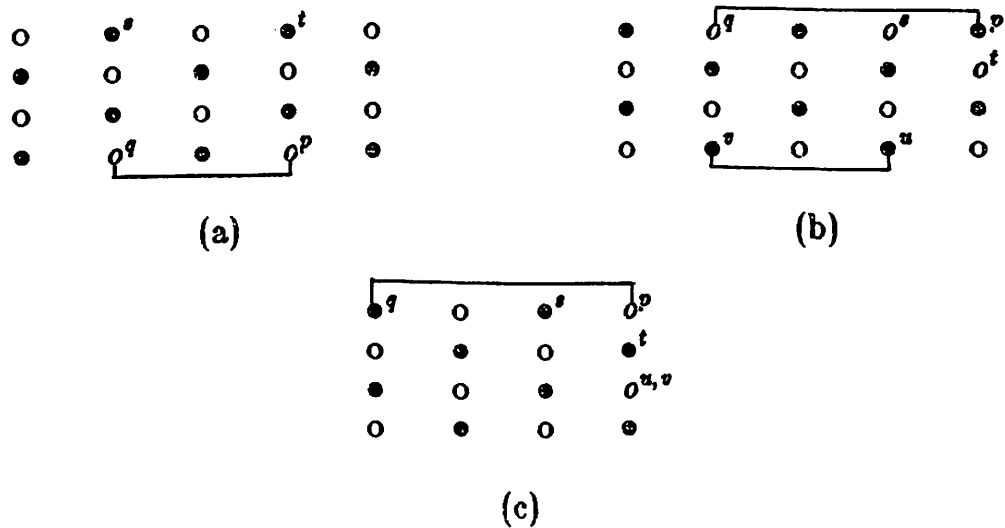


Figure 8.1. Unsolvability extra edge L-shaped problems.

Consider Step 6. This is solvable by Lemma 6.5. Step 7 is solvable by the same argument as given in the proof of Algorithm 2. $\square$

8.2. Small L-shaped Grid Graphs

Let (L^{1-3}, s, t) be the Hamiltonian path problem for L-shaped grid graphs without holes whose concave-vertex decomposition is smaller than size four. This section shows that whether or not L^{1-3} has a Hamiltonian path can be determined and a path can be found if one exists in linear time.

8.2.1. Preliminaries

Call the rectangles in the concave-vertex decomposition of an L-shaped grid graph A, B, and C as shown in Figure 8.2. Whereas Algorithm 2 was sufficient to solve (L^4, s, t) , the following observations indicate that a new approach is required to solve (L^{1-3}, s, t) . Algorithm 2 assumes that all acceptable problems are solvable. This is not the case for some of (L^{1-3}, s, t) . Figure 8.3 shows one example of an L-shaped grid graph problem which is acceptable but not solvable.

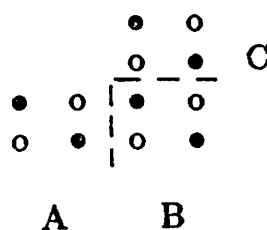


Figure 8.2. Concave-vertex decomposition of an L-shaped grid graph.

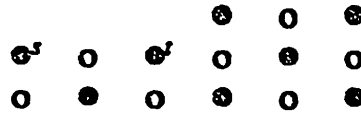


Figure 8.3 (L,s,t) is acceptable but not solvable.

Consider Figure 8.4. Algorithm 2 would have tried to solve the rectangle walk from A to B. However, for the problem in Figure 8.4 there is no solution to that rectangle walk. Recall that in Algorithm 2 the grouping graph is composed of nearest orthogonal neighbors and that the matching is obtained from the grouping graph. Instead suppose the matching is obtained from the decomposition graph. In the problem in Figure 8.4, if rectangles B and C were matched together, (call the resulting rectangle BC), then there is a solution to the walk from A to BC.

There are still other L-shaped problems that are solvable but no matching from the decomposition graph leads to a solution. See Figure 8.5 for an example. Fortunately, the number of problems in which this occurs is small. These problems will be called special case problems and handled separately. (L^{1-3},s,t) is a *special case* if the dimensions, (m,n) , of A are $(1,n)$, $n > 3$, and the dimensions of C are $(m,1)$, $m > 3$.

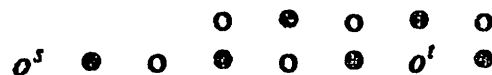


Figure 8.4. Cannot be solved by Algorithm 2.

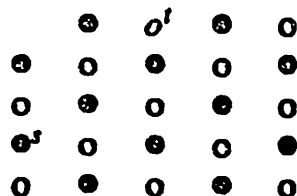


Figure 8.5. A special case problem.

The solution strategy for solving (L^{1-3}, s, t) which are not special cases is to try all possible matchings of the decomposition graph until one is found with a walk that solves (L^{1-3}, s, t) . If no such matching is found, as would be the case for the problem in Figure 8.3, then conclude that (L^{1-3}, s, t) has no Hamiltonian path. It remains to show that (L^{1-3}, s, t) is solvable if and only if there is a matching of the decomposition graph and a corresponding walk that solves it.

Note that the decomposition graph of an L-shaped grid graph, $D(L)$, always has three vertices so that a matching in $D(L)$ always matches two vertices and leaves one vertex unmatched. Thus a new decomposition corresponding to a matching contains two rectangles: AB and C or A and BC. Call the two rectangles in the new decomposition, X and Y. There are only two walks to consider, from X to Y (if s is in X and t is in Y) or from X to Y and back to X (if s and t are both in X). (L^{1-3}, s, t) is called *walkable* if there is some matching such that

- (i). s is in X, t is in Y, and there is an edge $\{p, q\}$ such that p is in X, q is in Y and (X, s, p) and (Y, t, q) are acceptable and not forbidden, or,

(ii). s and t are in X , (X,s,t) is acceptable and not forbidden and there are two edges $\{p,p'\}$ and $\{q,q'\}$ such that $\{p,q\}$ is on the solution of (X,s,t) , p' and q' are in Y and (Y,p',q') is acceptable and not forbidden, or

(iii). s and t are in X , X is not color compatible, there are two edges $\{p,p'\}$ and $\{q,q'\}$ such that there is a Hamiltonian path from s to t in X which goes through $\{p,q\}$ and (Y,p',q') is acceptable and not forbidden.

Lemma 8.2: Let (L^{1-3},s,t) be an acceptable problem that is not a special case. (L^{1-3},s,t) has a Hamiltonian path from s to t if and only if it is walkable.

Proof: Clearly if (L^{1-3},s,t) is walkable then it has a Hamiltonian path. It remains to show that if there is a Hamiltonian path then (L^{1-3},s,t) is walkable. The proof is by case analysis and can be found in Appendix B as Lemma B.4. $\square$

8.2.2. The Algorithm

Given (L^{1-3},s,t) , an acceptable L-shaped grid graph problem which is not a special case, the following algorithm reports a solution to (L^{1-3},s,t) or reports that (L^{1-3},s,t) has no solution.

Algorithm 3:

Step 1. Construct rectangles A, B and C using concave-vertex decomposition.

Step 2. Construct the decomposition graph and perform a matching. the walk.

Step 3. If (L^{1-3}, s, t) is walkable for the matching, solve it.

Step 4. If (L^{1-3}, s, t) is not walkable, go to step 2 and try another matching.

Step 5. If all matchings have been examined and (L^{1-3}, s, t) is not walkable for any then report that (L^{1-3}, s, t) has no solution.

8.2.3. Proof of Correctness

Lemma 3.2 shows that (L^{1-3}, s, t) has a Hamiltonian path if and only if it is walkable. It is required to show that the algorithm examines all possible walks before reaching Step 5 and concluding that there is no Hamiltonian path. There is only one possible walk for a particular matching. Step 4 ensures that all matchings are examined. Thus all possible walks are examined. $\square$

8.2.4. Timing Analysis

Clearly Steps 1 and 2 require linear time. In Step 3 it is necessary to determine whether (L^{1-3}, s, t) is walkable. This requires locating correct choices of p and q . Since each vertex is tested only once this must take linear time. It is required to show the number of times that the algorithm will iterate. Because L is composed of three rectangles in a fixed position there must be a constant number of matchings and walks. In fact there are only two possible matchings, each matching having one possible walk. Thus linear time is sufficient.

All that remains is to solve the special case problems.

Lemma 3.3: Whether or not there is a Hamiltonian path in a special case problem can be determined and a path can be found if one exists in linear time.

Proof: See Lemma B.5 in Appendix B. $\square$

Because acceptability can be determined in linear time, the following theorem can now be stated.

Theorem 4: Whether or not there is a Hamiltonian path in an L-shaped grid graph without holes can be determined and a path can be found if one exists in time linear in the number of vertices.

CHAPTER 9

Conclusions and Further Study

The purpose of this thesis was to examine the Hamiltonian path problem for non-rectangular grid graphs. The Hamiltonian path problem for non-rectangular grid graphs was interesting to study because there are few classes of restricted graphs for which the Hamiltonian path problem is polynomial.

9.1. Summary of Contributions

The first contribution of the thesis is a survey of the current knowledge of the Hamiltonian circuit and path problems. This provides a perspective from which to view the particular problem studied.

The contribution towards solving the Hamiltonian path problem in non-rectangular grid graphs is given in four theorems proved in the thesis:

Theorem 1: Whether or not there is a Hamiltonian path in a grid graph with or without holes that has an even decomposition of size four can be determined and a path can be found if one exists in time linear in the number of vertices.

Theorem 2: Whether or not there is a Hamiltonian path in a concave-vertex decomposable grid graph of size six with or without holes with a perfect or near perfect matching can be determined and a path can be found if

one exists in time linear in the number of vertices.

Theorem 3: Whether or not there is a Hamiltonian path in a pyramid grid graph without holes that has a concave-vertex decomposition of size six can be determined and a path can be found if one exists in time linear in the number of vertices.

Theorem 4: Whether or not there is a Hamiltonian path in an L-shaped grid graph without holes can be determined and a path can be found if one exists in time linear in the number of vertices.

It was a rather surprising result that Theorems 1 and 2 apply to grid graphs without holes as it was expected that the holes would make the problem NP-complete. This type of result contributes to the identification of the distinguishing characteristics of the P and NP-complete classes.

9.2. Further Work

Further study can be divided into two classes, further study on the Hamiltonian path problem in non-rectangular grid graphs and further study on related problems.

9.2.1. More Study on Non-Rectangular Grid Graphs

This thesis helps to solve the Hamiltonian path problem in non-rectangular grid graphs by solving the problem for four restricted classes of grid graphs that are more complex than rectangular grid graphs. However,

the problem remains open for general grid graphs without holes.

The thesis suggests some other questions which may eventually lead to a solution to this open problem. Theorem 2 states that perfect or near perfect matchings are a sufficient condition. Figure 9.1 shows that this condition is not necessary. The grouping graph of the graph shown has no near perfect or perfect matching yet it has a Hamiltonian path. Perhaps there is a more general restriction that is a necessary condition.

Being able to add all of the required extra edges was shown to a necessary condition in Chapter Five but this condition was never formally defined. For large grid graphs is acceptability and this new necessary condition sufficient?

It is shown by Theorem 4 that whether or not there is a Hamiltonian path in small L-shaped grid graphs can be determined in time linear in the number of vertices. The next step might be to consider the problem for small pyramid grid graphs.

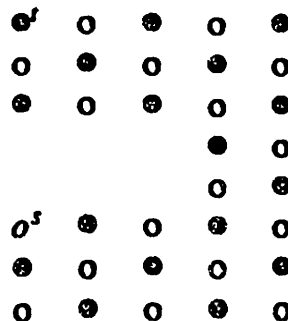


Figure 9.1. Near or perfect matchings are not necessary.

9.2.2. Related Problems

There are two variations of the Hamiltonian path problem. The first, which was considered in this thesis, is given a starting vertex and an ending vertex, is there a path which goes through all vertices. The second, not considered in this thesis, is, can a starting vertex and an ending vertex be found such that there is a Hamiltonian path.

A few observations can already be made concerning the second problem in grid graphs. Clearly, if the graph contains a circuit then s and t can be chosen as two consecutive vertices on the circuit. Luccio and Mugnai have shown that every rectangular grid graph with an even number of vertices has a Hamiltonian circuit [Luccio et al.] so the path problem is clearly solvable. If a grid graph has an even decomposition of size four then if s and t are chosen as two adjacent vertices on the boundary there is a path. For grid graphs whose grouping graph has a perfect matching choose s and t as two adjacent vertices on the boundary of a rectangle in the new decomposition. If the grouping graph of a grid graph has a near perfect matching then choose s and t as two adjacent vertices on the boundary of any but the unmatched rectangle in the new decomposition. For small L-shaped grid graphs the problem is more complex although from Lemma B.4 it is possible to enumerate all cases in which there is no Hamiltonian path and then choose s and t . In general grid graphs without holes the Hamiltonian path problem without specified endpoints remains open.

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Appendix A. Extra Edge Problems

Let (R,s,t) be a six by six or larger rectangular grid graph problem. A rectangular grid graph is said to have four sides corresponding to the edges of its representative polygon. Let E' be a set of edges such that both the endpoints of each edge are on the same side of the boundary of R . Let $R_e = R \cup E'$. (R_e,s,t) is called solvable if there is a Hamiltonian path from s to t that goes through all the edges in E' .

New conditions for R_e to be color compatible are required. Let the color of the elements of V_1 be 1 and the color of the elements of V_2 be -1 where V_1 and V_2 are the vertices of R_e .

Definition: Color compatibility. The Hamiltonian path problem (R_e,s,t) is color compatible if:

1. $(|V_1| = |V_2|)$ and the sum of the color of the endpoints of all edges in $E' +$ the color of $s +$ the color of $t = 0$, or
2. $(|V_1| = |V_2| + 1)$ and the sum of the color of the endpoints of all edges in $E' +$ the color of $s +$ the color of $t = 2$.

(R_e,s,t) is called acceptable if it is color compatible and connected.

Lemma A.1: Given (R,s,t) , a six by six or larger rectangular grid graph with an even number of vertices and s and t the same color, then there exists a set E' containing a single edge on any side of the boundary of R such that

(R_e, s, t) is solvable.

Proof: Let p and q be the endpoints of the edge to be in E' . Note that in order for (R_e, s, t) to be color compatible, p and q must be the same color and the opposite color of s and t . Because it is necessary to show only the existence of a set E' , it suffices to prove the result for a particular choice of p and q . If s and t are white then choose p and q as shown in Figure A.1(a) and as in Figure A.1(b) if s and t are black. In general, p and q are to the lower far right. Now show that for any choice of s and t there is a Hamiltonian path from s to t going through $\{p, q\}$. Note that if p and q were chosen to lie on a different side then the problem would be symmetrically equivalent to the one in Figure A.1. There are three cases to consider: R_e is even by even, even by odd or odd by even for each color of s and t . Figure A.1 shows the cases for R_e even by even for each color of s and t .

Case 1: R_e is even by even. Suppose p and q are black. (If p and q are white the proof is similar). Separate R_e vertically into k rectangles, each two

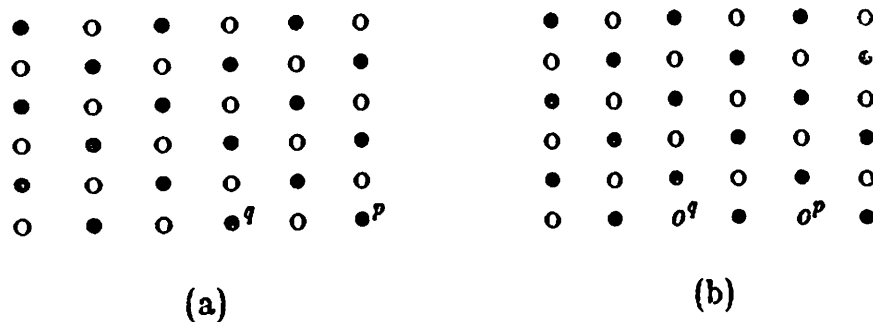


Figure A.1. Choosing p and q .

vertices wide (call them 2-rectangles) as shown in Figure A.2.

a) Suppose s is in one of the 2-rectangles 1 to $k-1$ and t is in the k th 2-rectangle. Form R_1 , the union of the 2-rectangles 1 to $k-1$ and call the k th 2-rectangle, R_2 . See Figure A.3(a) for an example when R_e is six by six. Clearly, (R_1, s, q) and (R_2, t, p) are both acceptable and not forbidden so (R_e, s, t) is solvable.

b) Suppose s and t are both in the k th 2-rectangle. Separate it horizontally into R_1 and R_2 such that s is in R_1 and t is in R_2 . This is always possible since s and t are the same color. See Figure A.3(b). Clearly, (R_1, s, p) is acceptable and not forbidden. Form R_3 , the union of the 2-rectangles 1 to $k-1$. Choose two adjacent vertices, u and v , such that u is in R_2 , v is in R_3 and (R_2, u, t) is acceptable and not forbidden. Clearly, (R_3, v, q) is acceptable and not forbidden so (R_e, s, t) is solvable.

c) Suppose s and t are in the 2-rectangles 1 to $k-1$. Assume that s is to the left of t . Strip (in the Itai et al. sense) off all the 2-rectangles to the left

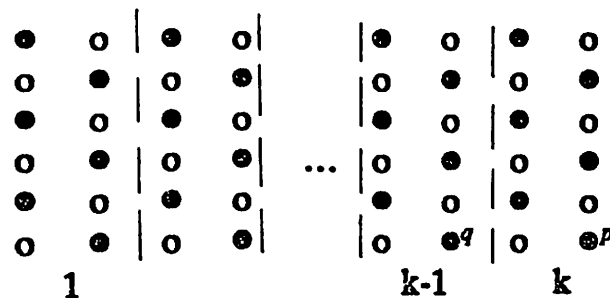


Figure A.2. Separated into 2-rectangles.

of s . If s and t are not in the same 2-rectangle then find an edge $\{x,y\}$ such that x is in the 2-rectangle containing s , R_i , y is in the next 2-rectangle, and (R_i, s, x) is acceptable and not forbidden. Continue in this manner until y is in the same 2-rectangle as t choosing $\{x,y\}$ so that y is more than one row away from t . See Figure A.3(c). Separate the 2-rectangle containing y and t into R_1 and R_2 such that $\{u, u'\}$ is an edge with u in R_2 , (R_2, y, u) acceptable and not forbidden, u' adjacent to u in the 2-rectangle in the right, $\{v, v'\}$ is an edge with v in R_1 , (R_1, t, v) acceptable and not forbidden and v' adjacent to v in the 2-rectangle to the right. Continue separating the 2-rectangle containing u' and v' until the $k-1$ 2-rectangle is reached. Separate the $k-1$ 2-rectangle into R_3 and R_4 such that (R_3, u', q) is acceptable and not forbidden and there is an edge $\{a, b\}$ such that a is in R_4 , (R_4, v', a) is acceptable and not forbidden, and b is adjacent to a in the k th rectangle, R_k . (R_k, b, p) is acceptable and not forbidden so (R_e, s, t) is solvable.

Case 2: R_e is even by odd. The construction in Case 1 applies.

Case 3: R_e is odd by even. Separate R_e vertically into k rectangles, $k-1$ of which are 2-rectangles and one of which is a 3-rectangle (a rectangle three vertices wide) where the k th rectangle is not the 3-rectangle, see Figure A.4(a). Since the 3-rectangle contains an even number of vertices, the constructions in Case 1(a) and (b) apply. Suppose s and t are in the 2-rectangles 1 to $k-1$. If neither s nor t is in the 3-rectangle then strip off the 3-rectangle and proceed as in Case 1(c). If s and t are both in the 3-rectangle, then

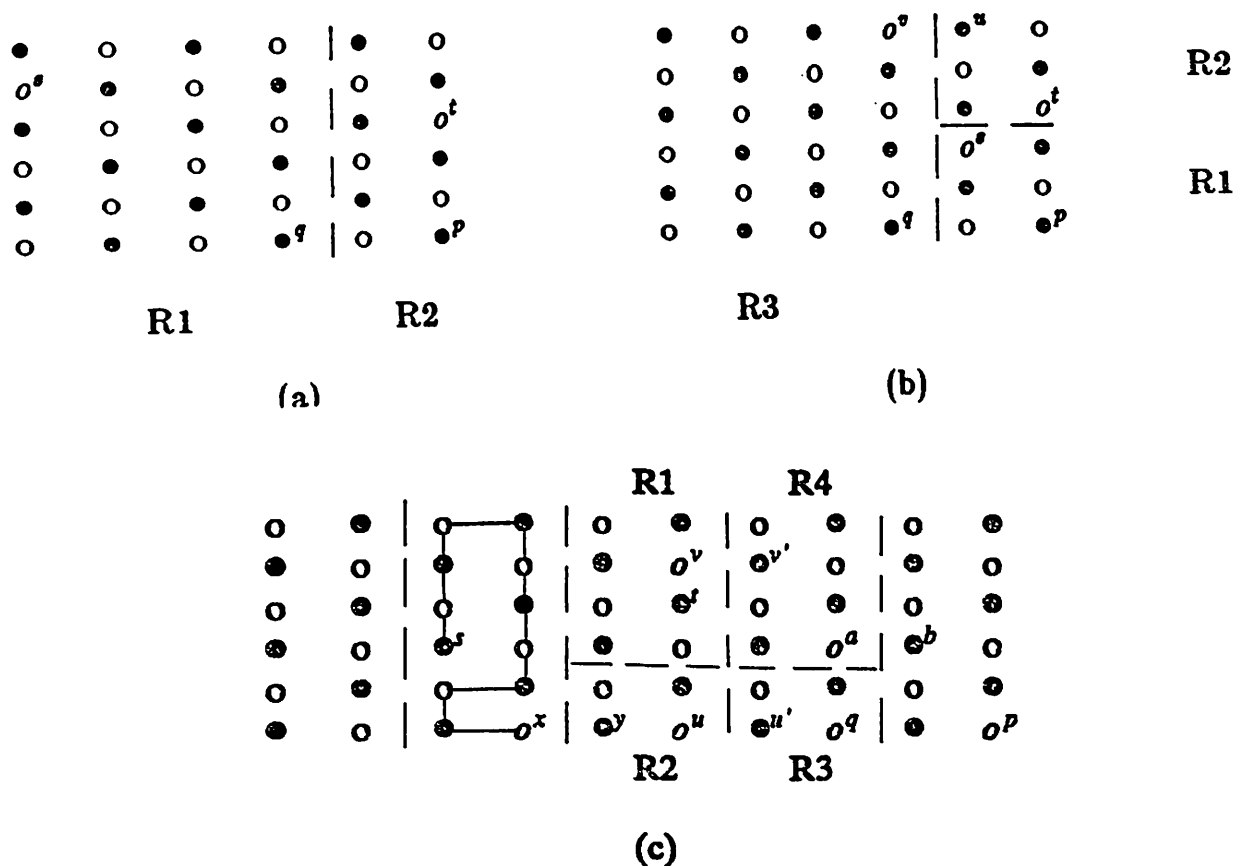
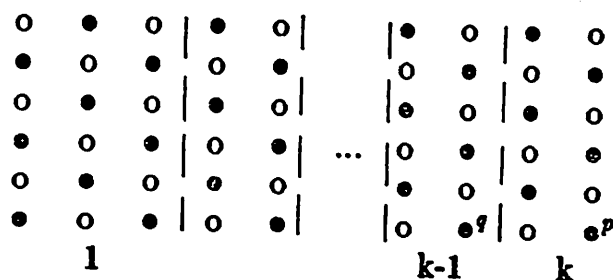


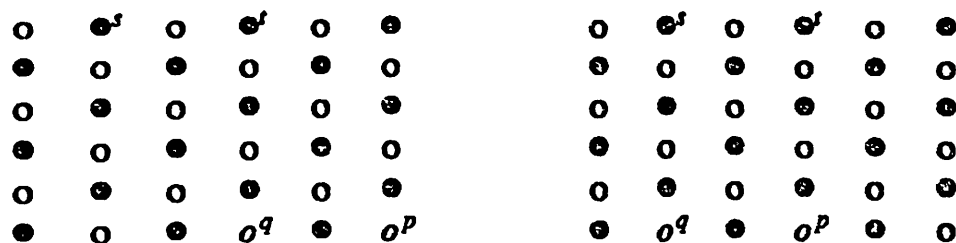
Figure A.3. Solving the 2-rectangles.

separate the 3-rectangle into R_1 and R_2 and find edges $\{u, u'\}$ and $\{v, v'\}$ as in Case 1(c). If s is in the 3-rectangle and t is not then there is only one situation in which it is not possible to find an edge $\{x, y\}$ as described in Case 1(c) and that situation is shown in Figure A.4(b). In this case choose a different $\{p, q\}$, as shown in Figure A.4(c), and solve as in Case 1(a). Note that this would not be possible if R were smaller than six by six. $\square$

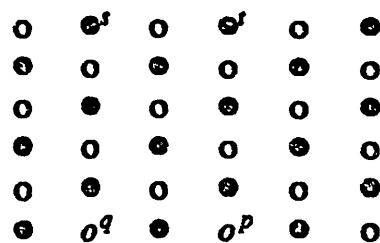
Lemma A.2: Given (R,s,t) , a six by six or larger rectangular grid graph problem with an even number of vertices and s and t disconnecting a corner vertex. There exists a set E' containing a single edge from the corner vertex to the vertex distance two from the corner on the boundary of R such that



(a)



(b)



(c)

Figure A.4. Separating into 2-rectangles and a 3-rectangle.

(R_e, s, t) is solvable.

Proof: Let p be the corner vertex and q the other end of the extra edge. Separate R_e into R_1 , R_2 and R_3 such that R_1 is the first two vertices in the first two rows of R_e , R_2 is what remains of the first two rows and R_3 is what remains of R_e , see Figure A.5.

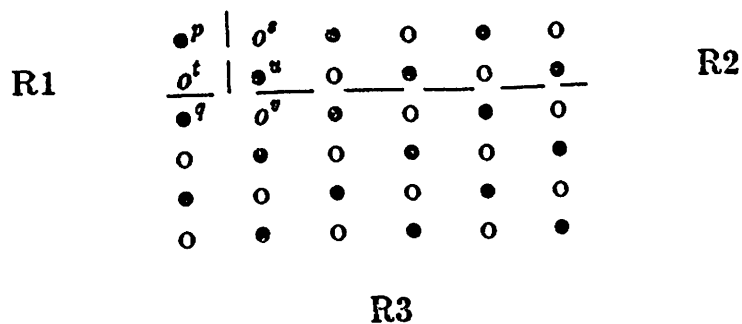


Figure A.5. Solving a disconnected graph.

Clearly, $(R1,p,t)$ is acceptable and not forbidden. Let u and v be the vertices shown in Figure A.5. Clearly, $(R2,s,u)$ and $(R3,q,v)$ are always acceptable and not forbidden so (R_e,s,t) is solvable. $\square$

Lemma A.3: Given (R,s,t) , a six by six or larger rectangular grid graph with an even number of vertices and s and t disconnecting a corner vertex. There exists a set E' consisting of an edge from the corner vertex to another vertex along either of the sides containing s or t with the opposite color and an edge on any side of the boundary of R that does not contain the other edge such that (R_e,s,t) is solvable.

Proof: Let p and q be the endpoints of the first edge where p is the disconnected corner vertex. Let u and v be the endpoints of the second edge. Note that in order for (R_e,s,t) to be color compatible, u and v must be the same color and the opposite color of s and t . It is sufficient to show that for a particular choice of p and q and a particular choice of u and v , (R_e,s,t) is solvable for all s and t . p is the corner vertex and choose q to be the vertex three to the left as shown in Figure A.6. Since u and v can lie on any side other than that containing p and q , there are three cases: u and v are chosen as far to the bottom right hand corner as possible in Case 1, as far to the bottom left hand corner as possible in Case 2 and as far to the top right in Case 3, see Figure A.6. For each of the three cases there are three choices for the dimensions of R_e : R_e is even by even, even by odd or odd by even. In all cases, these three possibilities can be treated similarly.

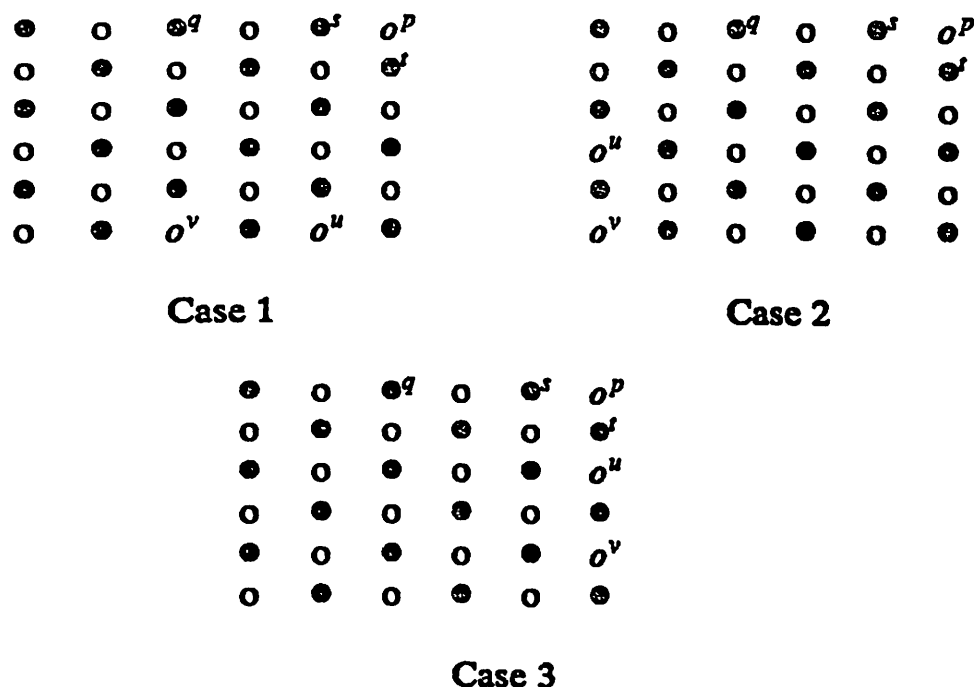


Figure A.6. Choosing $p, q, u,$ and v .

Case 1: u and v are on the last row. Separate R_c into R_1, R_2 and R_3 . R_1 is all but the two columns on the far right. R_2 and R_3 are the two rectangles formed by separating the two columns on the far right between s and t . An example is shown in Figure A.7. Clearly, (R_1, v, q) , (R_2, s, p) and (R_3, t, u) are all acceptable and not forbidden so (R_c, s, t) is solvable. Note that if R is smaller than a six by six rectangle then (R_1, v, q) may be forbidden.

Case 2: u and v are on the first column. Separate R_c into R_1, R_2, R_3 and R_4 . R_1 is the rectangle containing s and p . R_2 is the remainder of the rightmost two columns of R_c . R_3 is the bottom two rows of what remains. R_4 is the remainder. An example is shown in Figure A.8. Clearly, (R_1, s, p)

•	o	• ^q	o		• ^s	o ^p	R2
o	•	o	•		o	• ^t	
•	o	•	o		•	o	
o	•	o	•		o	•	R3
•	o	•	o		•	o	
o	•	o ^v	•		o ^u	•	
R1							

Figure A.7. Solving Case 1.

is acceptable and not forbidden. Choose two vertices, y and z , such that y is in $R2$ and $(R2, t, y)$ is acceptable and not forbidden and z is adjacent to y in $R3$ and $(R3, v, z)$ is acceptable and not forbidden. Clearly, $(R4, u, q)$ is acceptable and not forbidden. Thus (R_e, s, t) is solvable.

Case 3: u and v are in the last column. Separate R_e into $R1$, $R2$, $R3$ and $R4$. $R1$ is the rectangle containing s and p . $R2$ is the next three rows of what remains of the rightmost two columns of R_e . $R3$ is what remains of the rightmost two columns. $R4$ is the remainder. An example is shown in Figure A.9. Clearly, $(R1, s, p)$ and $(R2, t, u)$ are acceptable and not forbidden. Choose two vertices, y and z , such that y is in $R3$ and $(R3, v, y)$ is acceptable

	•	o	• ^q	o		• ^s	o ^p	R1
R4	o	•	o	•		o	• ^t	
	•	o	•	o		•	o	
	o ^u	•	o	•		o	•	R2
	•	o	•	o		•	o	
R3	o ^v	•	o	• ^z		o ^y	•	

Figure A.8. Solving Case 2.

and not forbidden and z is adjacent to y in R_4 . Clearly, (R_4, q, z) is acceptable and not forbidden. Thus (R_c) is solvable. Note that if R is smaller than six by six it would not be possible to choose an edge $\{u, v\}$. $\square$

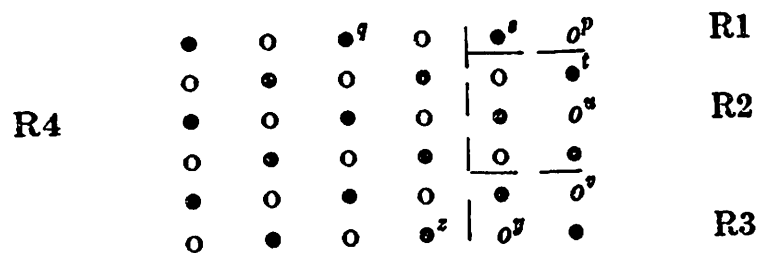


Figure A.9. Solving Case 3.

Appendix B. Small Problems

The following three lemmas are used in the proof of Lemma B.4.

Lemma B.1: Given (R,s,t) , a four by four or larger rectangular grid graph with an even number of vertices and s and t the same color but not the color of the right corner vertex, then there exists a set E' containing a single edge, $\{p,q\}$, on the upper side of R with p the right corner vertex (see Figure B.1), such that (R_e,s,t) is solvable.

Proof: Note that p and q cannot coincide with s and t because they are colored differently. This is actually a special case of Lemma A.1 which solves the problem for six by six or larger rectangular grid graphs. That it holds for four by four rectangular grid graphs follows from the proof of Lemma 8.1. $\square$

Lemma B.2: Given (R,s,t) , a five by five or larger rectangular grid graph with an odd number of vertices and s and t different colors, then there

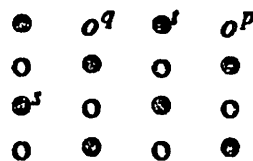


Figure B.1. Problem defined in Lemma B.1.

exists a set E' containing a single edge, $\{p,q\}$, on the upper side of R with p the right corner vertex (see Figure B.2), such that (R_e,s,t) is solvable.

Proof: Assume neither p nor q coincides with s or t . Separate R into R_1 and R_2 such that R_1 contains all rows from the left up to and including the row containing q as shown in Figure B.2(a). If s is in R_1 and t is in R_2 and (R_1,s,q) and (R_2,t,p) are color compatible then they are acceptable and not forbidden and (R_e,s,t) is solvable. If (R_1,s,q) is not color compatible then separate as shown in Figure B.2(b) and solve (L_1,s,u) (by Lemma B.4). (R_3,v,t) is solvable with $\{y,z\}$ an edge on the solution. (R_4,y',z') has a solution going through $\{p,q\}$.

Suppose s and t are in R_2 . If they are not in the first two rows then separate as shown in Figure B.2(c). Choose $\{p,q\}$ as shown and solve as above. If they are in the top two rows solve as Figure B.2(d) by choosing an edge $\{u,v\}$ and solving the two L-shaped problems and the rectangular problem shown or, if t is above s , the three rectangular problems.

Suppose s and t are in R_1 . If color compatible rectangles result, separate as shown in Figure B.2(e). Otherwise, separate as in Figure B.2(f). $\{u,v\}$ is an edge on the solution of (R_1,t,y) and $\{u',v'\}$ is an edge on the solution of (R_2,p,q) .

Note that p and q are the same color so it is not possible for both s to coincide with p and t with q (or vice versa). If s or t corresponds to q then choose the other color compatible vertex as q . If s or t corresponds to p

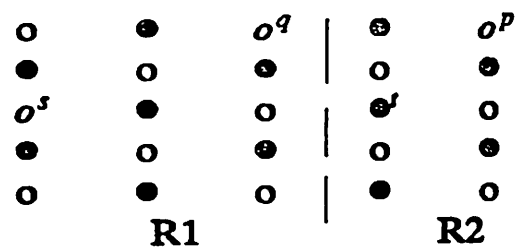
(assume s) then choose q to be in the top left corner of R . If t is not in the top row then separate into R_1 , R_2 , and R_3 as shown in Figure B.2(g) and choose $\{u, u'\}$ as shown. Clearly, (R_1, q, u) and (R_3, t, u') are acceptable and not forbidden. If t is in the top row then separate into R_1 , R_2 , R_3 and R_4 as shown in Figure B.2(h) and choose $\{u, u'\}$ and $\{v, v'\}$ as shown. Clearly, (R_2, t, u) , (R_3, q, v) and (R_4, v', u') are acceptable and not forbidden. $\square$

Lemma B'3: Given (R, s, t) , a four by six or larger rectangular grid graph with an even number of vertices and s and t different colors. If R can be separated into R_1 and R_2 such that (R_1, s, t) consists of the leftmost two or three columns and is not acceptable or is forbidden then there exists a set E' containing a single edge on the upper side of R with p the right corner vertex of R_2 and q the left corner vertex of R_2 such that (R, s, t) is solvable.

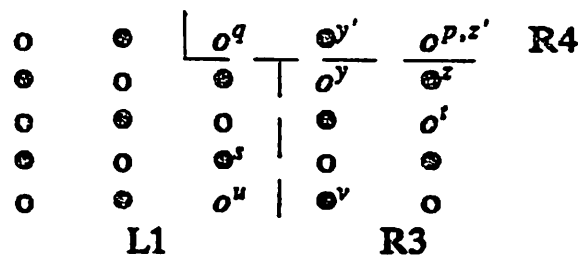
Proof: There are two cases; (R_1, s, t) can be a disconnected 2-rectangle or a forbidden 3-rectangle.

Case 1: (R_1, s, t) is a disconnected 2-rectangle. If s and t are not in the top two rows, separate R into R_3 , R_4 and R_5 such that R_3 consists of the first two rows up to the first three columns, R_4 is the remainder of the first two rows and R_5 is what remains, see Figure B.3(a). Solve (R_3, q, u') , (R_4, p, v') and (R_5, s, t) where $\{p, q\}$, $\{u, u'\}$ and $\{v, v'\}$ are chosen as shown.

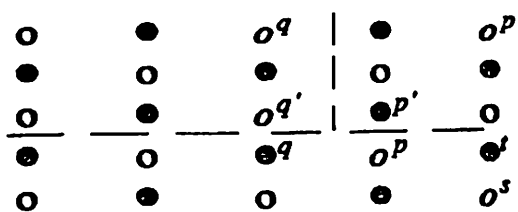
If s and t are in the top two rows then, since (R_1, s, t) is disconnected, s and t must be in the second row. Separate R into L_1 , L_2 and R_3 as shown in Figure B.3(b). Clearly, (L_1, s, q) is solvable. (L_2, t, p) has a solution which



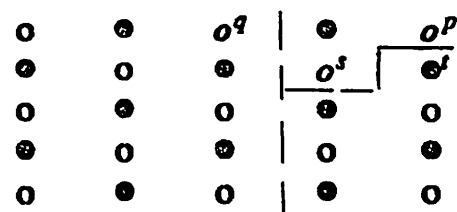
(a)



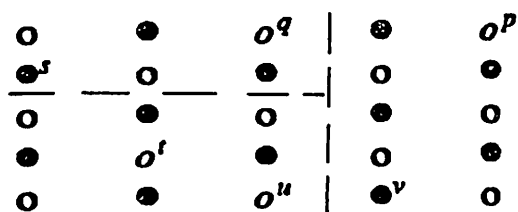
(b)



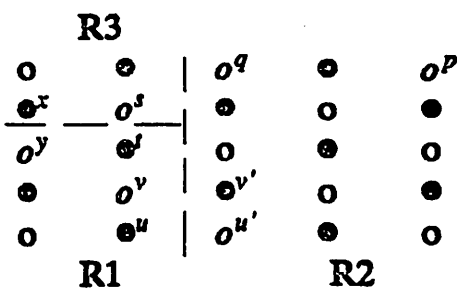
(c)



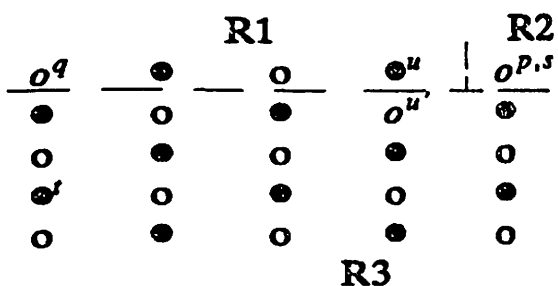
(d)



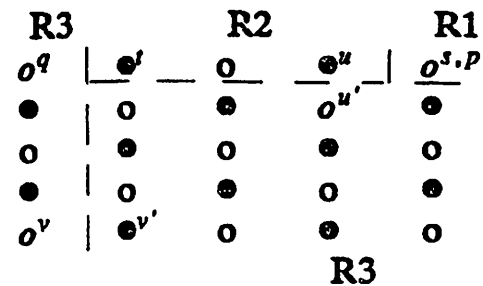
(e)



(f)



(g)



(h)

Figure B.2. Problem defined in Lemma B.2.

goes through $\{u, v\}$, and $(R3, u', v')$ is acceptable and not forbidden.

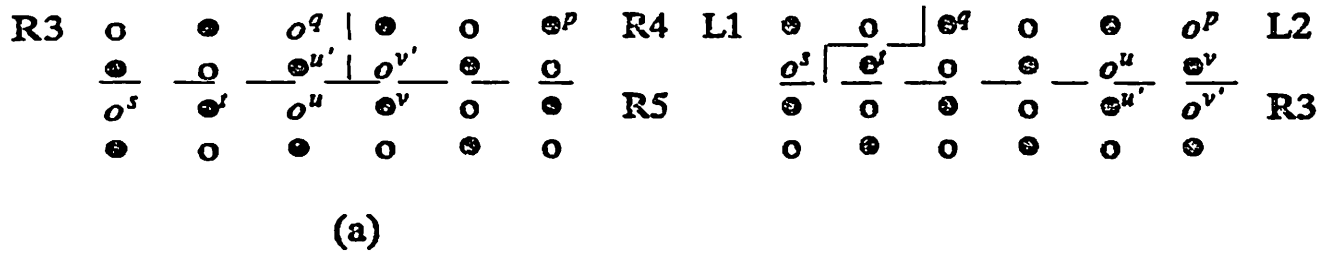
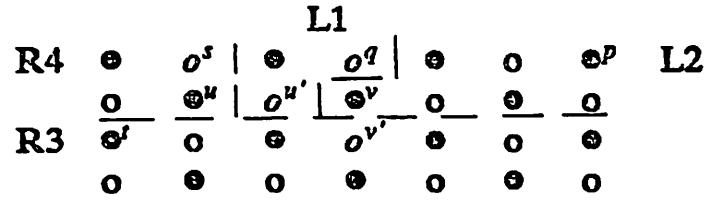


Figure B.3. Disconnected 2-rectangle.

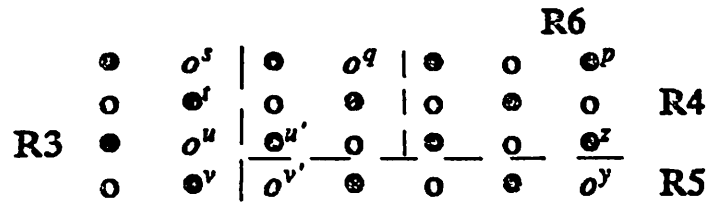
Case 2: $(R1, s, t)$ is a forbidden 3-rectangle. Separate R into $R3$, $R4$, $L1$, and $L2$ such that $R3$ contains all of the rows from the bottom up to and including t , $R4$ is the columns up to and including s of what remains and $L1$ and $L2$ are as shown. If neither $R3$ nor $R4$ contains only one row choose $\{u, u'\}$ and $\{v, v'\}$ as shown in Figure B.4(a) and solve $(R3, t, v')$, $(R4, s, u)$, $(L1, u', q)$ and $(L2, v, p)$. Otherwise, s and t must be as shown in Figure B.4(b) or in a similar position but in the last two rows. Separate R into $R3$, $R4$, $R5$ and $R6$ such that $R3$ is the first two columns, $\{u, v\}$ is an edge on the solution of $(R3, s, t)$, $R5$ is what remains of the last rows up to v' , $R6$ is what remains of the top rows including column q and $R4$ is what remains. Solve $(R5, v', y)$, $(R4, z, p)$ and $(R6, u', q)$. $\square$

Lemma B.4: If (L^{1-3}, s, t) is acceptable and not a special case then it has a Hamiltonian path if (L^{1-3}, s, t) is walkable.

Proof: The proof is by cases. For each case it is shown why (L^{1-3}, s, t) is walkable and if this is not possible it is shown that there is no Hamiltonian path. To show (L^{1-3}, s, t) walkable, a matching of the decomposition graph



(a)



(b)

Figure B.4. Forbidden 3-rectangle.

must be specified. If A and B are the rectangles in L corresponding to the matched vertices in the decomposition graph then the rectangle formed by their union will be denoted AB and A and B will be said to be matched. Depending on the location of s and t, edges must be found and sub-problems shown acceptable and not forbidden.

First some claims will be made about sufficient conditions for the acceptability and non-forbiddenness of rectangular problems. All of these claims follow directly from the definition of acceptability and forbiddenness for rectangular grid graphs found in Chapter Three.

Claim 1: If (R,s,t) is a color compatible rectangle of dimensions (m,n) where $m > 1$ and $n > 1$, then if s is in a corner, (R,s,t) is acceptable and not forbidden.

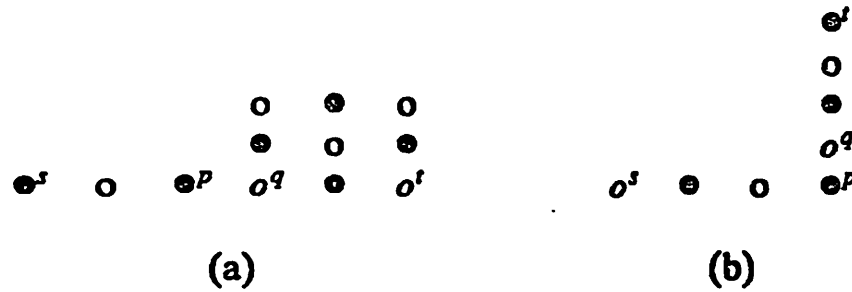
Claim 2: If (R,s,t) is a color compatible rectangle of dimensions (m,n) where $m > 1$ and $n > 1$, then if s and t are adjacent to one another and lie on the boundary of R , (R,s,t) is acceptable and not forbidden.

Claim 3: If (R,s,t) is a color compatible rectangle of dimensions (m,n) where $m > 3$ and $n > 3$, then (R,s,t) is acceptable and not forbidden.

A is called a 1-rectangle if it has dimensions, n by 1. C is called a 1-rectangle if it has dimensions, 1 by n . Similar definitions apply for 2-rectangle and 3-rectangle.

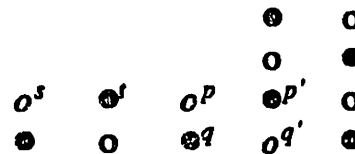
Case 1: L^{1-3} contains a 1-rectangle. In L^{1-3} , either A or C or both are 1-rectangles. Suppose A is a 1-rectangle. Match B and C . Since (L^{1-3},s,t) is acceptable it is connected so s must be at the extreme end of A and t must be in either B or C . Note that if C is also a 1-rectangle, then t must be at the extreme end of C . In order for (L^{1-3},s,t) to be walkable, it is necessary to find an edge $\{p,q\}$ such that p is in A , q is in BC , and (A,s,p) and (BC,q,t) are acceptable and not forbidden. There is only one candidate which is shown in Figure B.5(a). Since (L^{1-3},s,t) is connected, t cannot coincide with q . If C is a 1-rectangle as in Figure B.5(b), then (BC,q,t) must be acceptable and not forbidden and if not then (EC,q,t) is acceptable and not forbidden by Claim 1.

Case 2: L_{1-3} contains a 2-rectangle. Either A is a 2-rectangle or A and C are both are 2-rectangles. Match B and C .

Figure B.5. A is a 1-rectangle.

(i). s and t are in A . Since (L^{1-3}, s, t) is acceptable, (A, s, t) must be connected. Suppose (A, s, t) is also color compatible. In order to show (L^{1-3}, s, t) walkable it is necessary to find edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the solution to (A, s, t) and (BC, p', q') is acceptable and not forbidden. Choose the edges as in Figure B.6. Since q' is in the corner, (BC, p', q') is acceptable and not forbidden by Claim 1. $\{p, q\}$ lies on the solution to (A, s, t) , even if s or t coincides with p or q , unless both s and t coincide with p and q . However, this is not possible since (L^2, s, t) is connected.

Suppose (A, s, t) is not color compatible as shown in Figure B.7. In order to show (L^{1-3}, s, t) walkable it is necessary to find edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the solution to (A, s, t) and (BC, p', q') is acceptable and not forbidden. In order for (EC, p', q') to be color compatible, p and q must

Figure B.6. A is a 2-rectangle, (A, s, t) color compatible.

be the same color so there is no possible choice of p and q and thus (L^{1-3}, s, t) is not walkable. Since any Hamiltonian path which solves (R, s, t) must cover the vertices in BC the path must leave A and return again. There are only two vertices from which to leave and return no path can leave return and then leave and return again. Thus the problem must be solved as two rectangle problems but these problems cannot be color compatible. Thus, there is no Hamiltonian path which solves (L^{1-3}, s, t) .

(ii). s is in A and t is in B or C . It is necessary to find an edge $\{p, q\}$ such that (A, s, p) and (BC, q, t) are acceptable and not forbidden. There is one and only one choice of $\{p, q\}$ which makes (A, s, p) color compatible, (see Figure B.8(a)), and since it must be connected (A, s, p) is acceptable and not forbidden.

Consider (BC, q, t) . (BC, q, t) must be color compatible. However, it need not be acceptable or not forbidden. Figure B.8(b) shows an example in which (BC, q, t) is disconnected, (c) shows an example in which q coincides with t , and (d) and (e) show examples in which (BC, q, t) is forbidden. Thus, in each of these cases (L^{1-3}, s, t) is not walkable. Note that any Hamiltonian

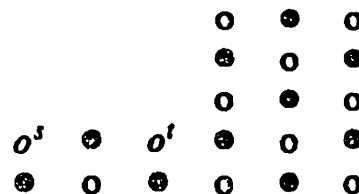


Figure B.7. A is a 2-rectangle, (A, s, t) not color compatible.

path must leave A and that once it has left it cannot return and leave again since there are only two possible vertices from which to leave. In fact, there is only one possible vertex, the color compatible one. Thus if the exit vertex coincides with t (as in Figure B.8(c)) or if the problem which remains after A has been exited is a forbidden problem (as in Figure B.8(d) and (e)) then there is no Hamiltonian path.

(iii). s and t are in B or C . It is necessary to find edges $\{p, p'\}$ and $\{q, q'\}$ such that there is a solution to (BC, s, t) which goes through $\{p, q\}$. There is only one possible choice of $\{p, q\}$, that shown in Figure B.9(a). (A, p', q') is acceptable and not forbidden by Claim 1.

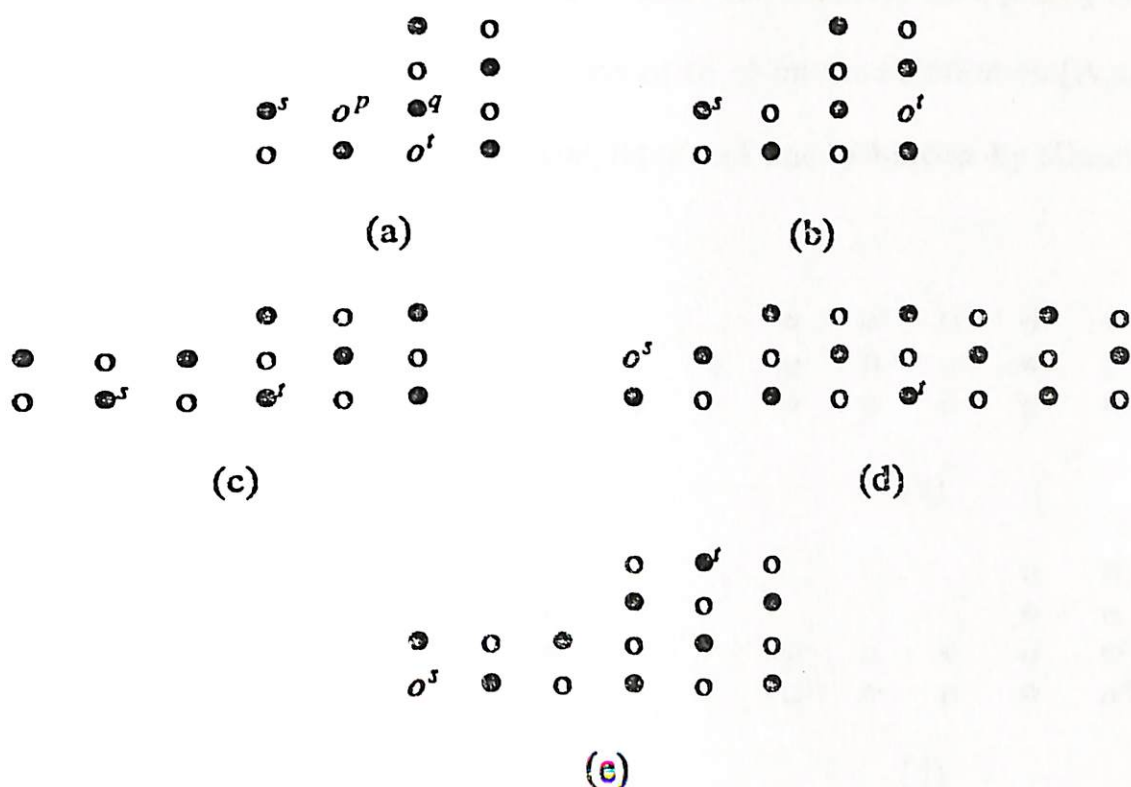


Figure B.8. A is a 2-rectangle, s in A , t in B or C .

Consider (BC, s, t) . (BC, s, t) must be color compatible. However, there need not be a path which goes through $\{p, q\}$. Figure B.9(b), (c) and (d) are all examples of problems which do not have such a path. However, note that for each of these cases there is no Hamiltonian path.

Case 3: L^3 contains a 3-rectangle. Either A is a 3-rectangle or A and C are both 3-rectangles.

(i) s and t are in A . Suppose (A, s, t) is color compatible. (A, s, t) must be connected. Match B and C . It is necessary to find edges $\{p, p'\}$ and $\{q, q'\}$. Note that BC contains an even number of vertices so p' and q' must be different colors. If (A, s, t) is solvable then there two possible positions for $\{p, q\}$. (Note that at most one of s and t can coincide with p or q and even if this is the case there is still a choice of $\{p, q\}$ on the solution to (A, s, t) .) In either case (BC, p', q') must be acceptable and not forbidden by Claim 2. An

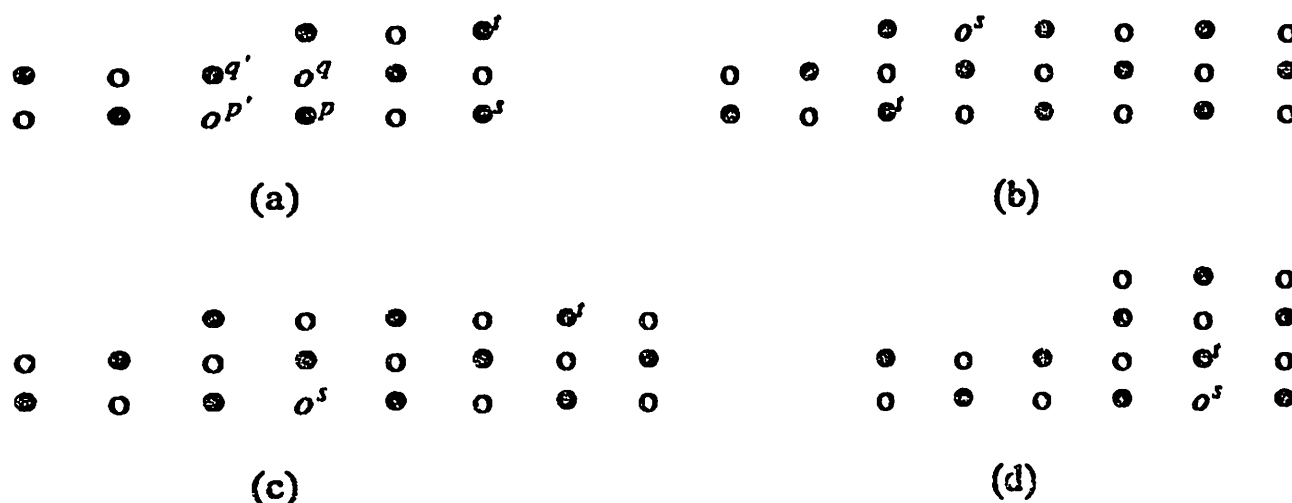


Figure B.9. A is a 2-rectangle, s and t in B or C .

example is shown in Figure B.10(a). Figure B.10(b) shows an example of (A,s,t) having no solution; it is forbidden. However, this problem has no Hamiltonian path.

Suppose (A,s,t) is not color compatible. Match A and B. It is necessary to find edges $\{p,p'\}$ and $\{q,q'\}$. Note that C contains an even number of vertices so p' and q' must be different colors. If (AB,s,t) is solvable then there are two possible positions for $\{p,q\}$. In either case (C,p',q') must be acceptable and not forbidden by Claim 2. An example is shown in Figure B.11(a). Figure B.11(b) shows an example of (AB,s,t) having no solution; it is forbidden. However, this problem has no Hamiltonian path.

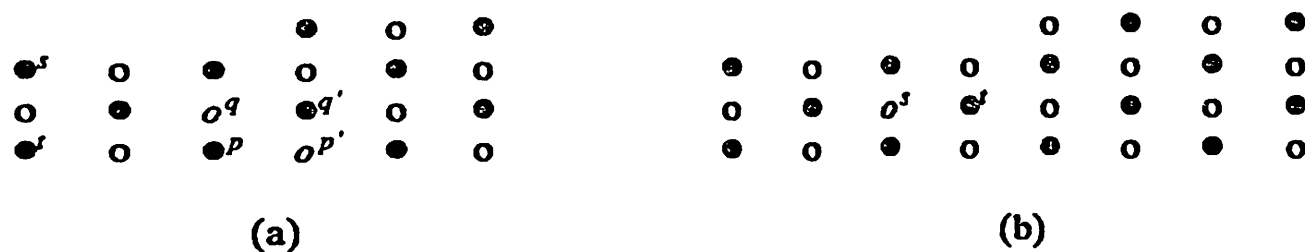


Figure B.10. A is a 3-rectangle, (A,s,t) is color compatible.

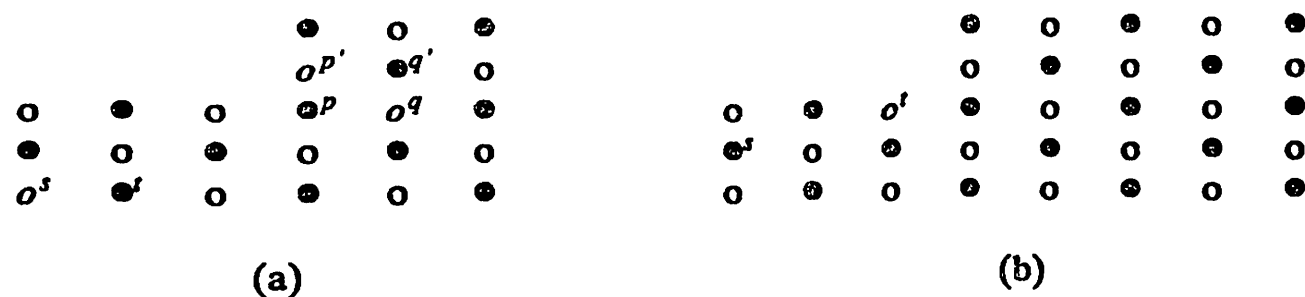


Figure B.11. A is a 3-rectangle, (A,s,t) is not color compatible.

(ii) s in A , t in B . Match A and B . Suppose (AB, s, t) is color compatible. If (AB, s, t) is acceptable and not forbidden then there exists edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the solution to (AB, s, t) . (C, p', q') must be connected and color compatible and by Claim 2, solvable. Figure B.12(a) shows one example. If (AB, s, t) is not acceptable or is forbidden, as in Figure B.12(b), then it must be forbidden. However, there is no Hamiltonian path for this problem.

Suppose (AB, s, t) is not color compatible. Match B and C . Unless s is positioned as in Figure B.13(a) or (b) then there exists an edge $\{p, q\}$ such that (A, s, p) and (BC, q, t) are acceptable and not forbidden. Note that there is no Hamiltonian path in the examples in Figure B.13(a) and (b).

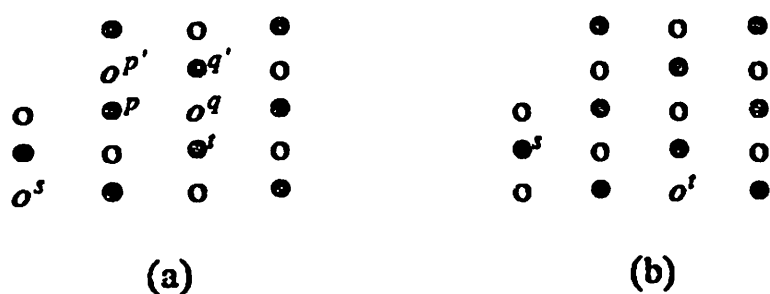


Figure B.12. A is a 3-rectangle, s in A , t in B .

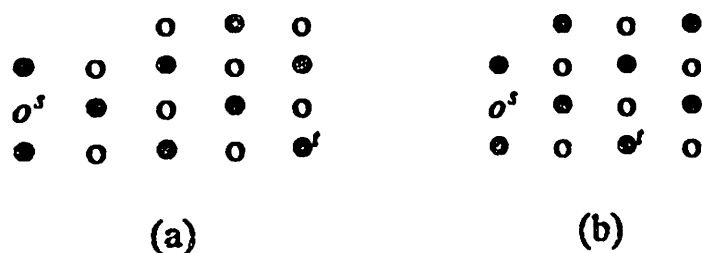


Figure B.13. A is a 3-rectangle, s in A , t in B .

(iii). s in A t in C . Match B and C . If no edge $\{p,q\}$ can be found such that (A,s,p) is acceptable and not forbidden, as in Figure B.14(a), then there is no Hamiltonian path. Otherwise, (C,q,t) is acceptable and not forbidden by Claim 3 unless it is a forbidden. Figure B.14(b) shows this case. However, (b) has no Hamiltonian path.

(iv). s and t in B or C . If C is a 3-rectangle solve as (ii). Otherwise, match B and C . If (BC,s,t) is color compatible then it is solvable by Claim 3. Choose edges $\{p,p'\}$ and $\{q,q'\}$. A must contain an even number of vertices so (A,p',q') is acceptable and not forbidden by Claim 1. If A contains an odd number of vertices and if (BC,s,t) is not color compatible as in Figure B.15(a) and (b) then (BC,s,t) has a solution using edge $\{p,q\}$, chosen as shown, by Lemma B.1 and B.2.

Case 4: L^{1-3} contains an n by 1 rectangle, $n > 3$. If both A and B are n by 1 rectangles then (L^{1-3},s,t) is a special case. Suppose A is an n by 1 rectangle.

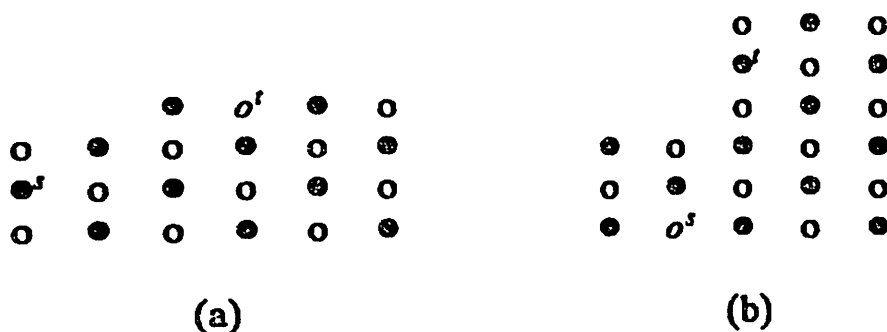


Figure B.14. A is a 3-rectangle, s in A , t in C .

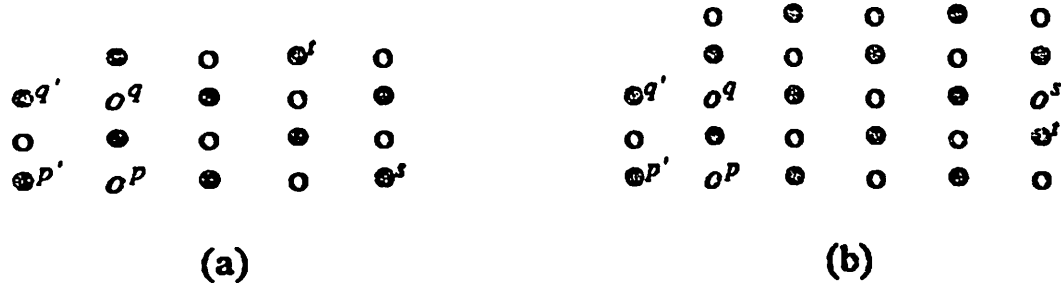


Figure B.15. A is a 3-rectangle, s and t in B or C.

(i). s and t are in A or B. Match A and B. Suppose (AB, s, t) is color compatible as in Figure B.16(a). (AB, s, t) is connected. By Claim 3 (AB, s, t) is solvable. There exists edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the solution to (AB, s, t) . (C, p', q') must be connected and color compatible and by Claim 2, solvable.

Suppose (AB, s, t) is not color compatible as in Figure B.16(b). In this case, p and q will not be adjacent. (AB, s, t) has a solution which goes through $\{p, q\}$ by Lemma B.1. (C, p', q') is solvable by Claim 1.

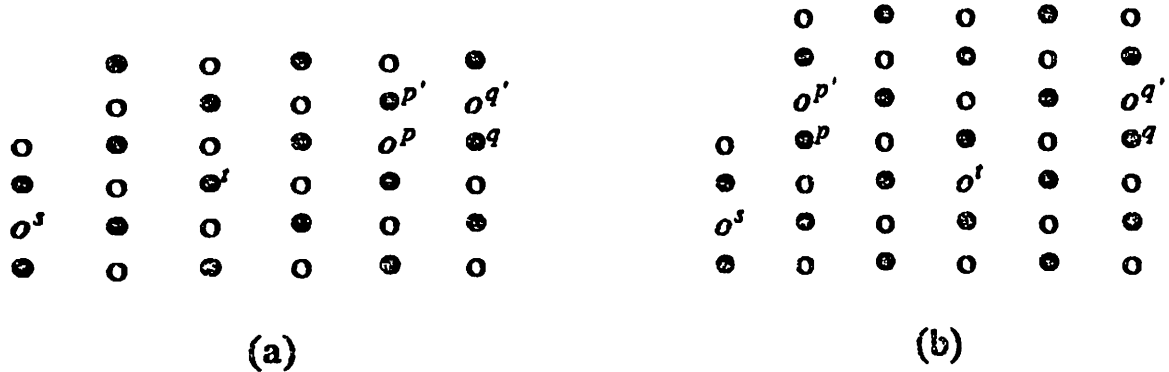


Figure B.16. A is an n by 1 rectangle, s and t in A or B.

(ii). s in A or B , t in C . Match A and B . It is necessary to find an edge $\{p,q\}$ such that (AB,s,p) and (C,q,t) are acceptable and not forbidden. Choose $\{p,q\}$ so that (AB,s,p) is color compatible and does not coincide with s . By Claim 3, (AB,s,p) is acceptable and not forbidden. $\{p,q\}$ can always be chosen so that (C,q,t) is acceptable and not forbidden. If C is an n by 2 rectangle then choose so as not to make (C,q,t) disconnected as shown in Figure B.17(a). If C is an n by 3 rectangle then choose so that (C,q,t) is not forbidden as shown in Figure B.17(b).

(iii). s and t in C . Suppose (C,s,t) is acceptable and not forbidden. Match A and B . There are edges $\{p,p'\}$ and $\{q,q'\}$ such that $\{p,q\}$ is on the solution to (C,s,t) . (AB,p',q') is acceptable and not forbidden by Claim 3. Suppose (C,s,t) is not acceptable or is forbidden. There are three possibilities: (C,s,t) is not color compatible as in Figure B.18(a), (C,s,t) is disconnected as in Figure B.18(b) and (c), or (C,s,t) is forbidden as in Figure

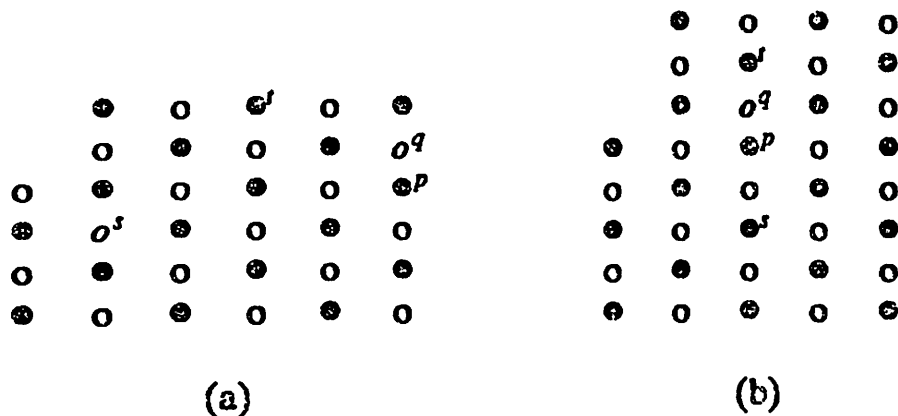


Figure B.17. A is an n by 1 rectangle, s in A or B , t in C .

B.18(d). Match B and C. There is only one choice for $\{p, p'\}$ and $\{q, q'\}$. Clearly, (A, p', q') is acceptable and not forbidden. That (EC, s, t) in Figure B.18(a) has a solution which uses $\{p, q\}$ is shown in Lemma B.1, in Figure B.18(b) and (d) in Lemma B.3, and in Figure B.18(c) in Lemma B.2.

Case 5: L^{1-3} contains an n by 2 rectangle, $n > 3$.

(i). s and t are in A or B. Match A and B. If (AB, s, t) is color compatible then it must be solvable by Claim 3. There exists edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the solution of (AB, s, t) and (C, p', q') is acceptable and not forbidden. Suppose (AB, s, t) is not color compatible. Choose $\{p, p'\}$ and $\{q, q'\}$ as in Figure B.19, then it is shown that there is a solution to (AB, s, t)

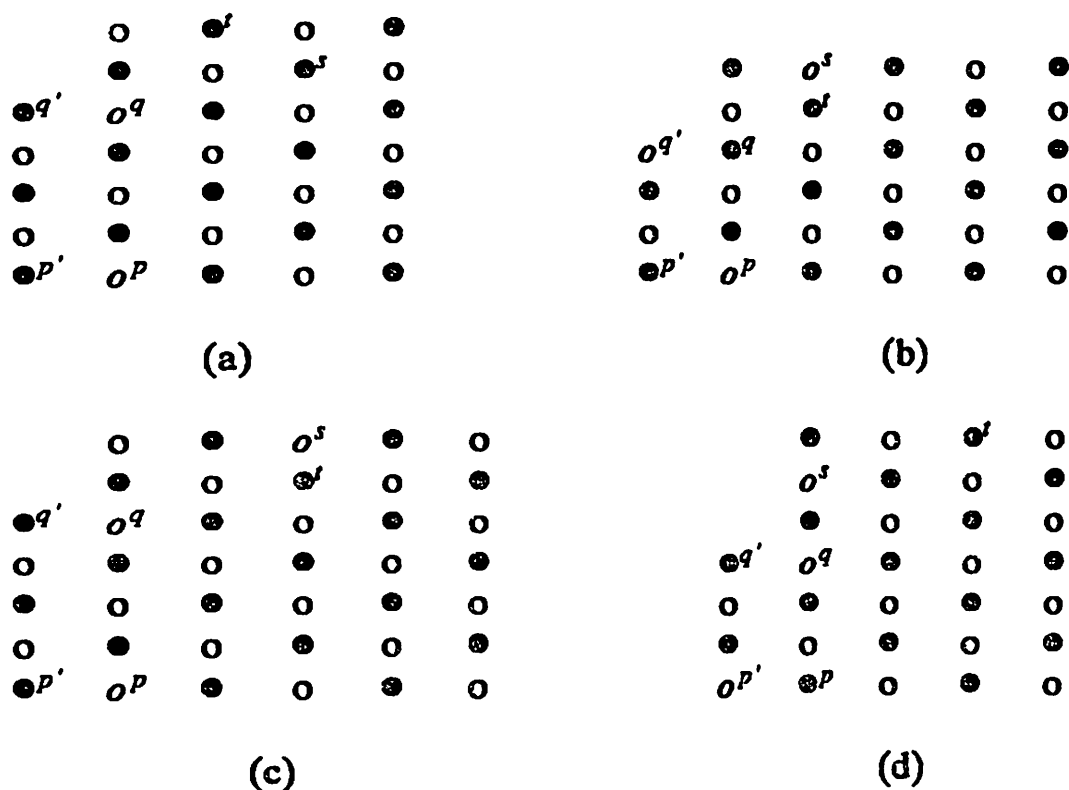


Figure B.18. A is a 1 by 3 rectangle, s and t in C.

which goes through $\{p,q\}$ in Lemma B.2. (C,p',q') is acceptable and not forbidden by Claim 1.

(ii) s and t are in B or C . Match B and C . (BC,s,t) must be color compatible because A has an even number of vertices. (BC,s,t) is solvable by Claim 3. There exist edges $\{p,p'\}$ and $\{q,q'\}$ such that (A,p',q') is acceptable and not forbidden by Claim 2.

(iii). s is in A and t is in C . Match B and C . There is always a choice of $\{p,q\}$ such that (A,s,p) and (BC,q,t) are acceptable and not forbidden. Figure B.20 shows one example.

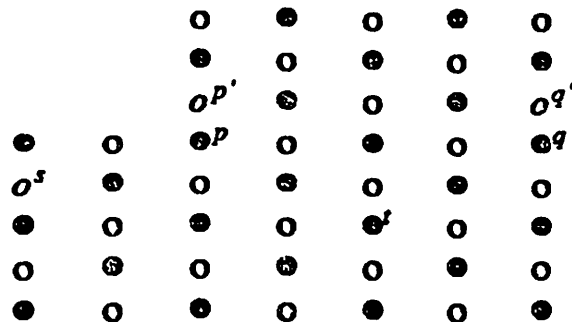


Figure B.19. A is an n by 2 rectangle, s and t are in A or B .

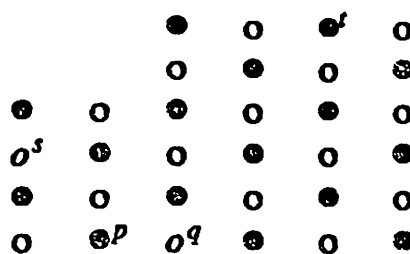


Figure B.20. A is an n by 2 rectangle, s is in A , t is in C .

Case 6: L^{1-3} contains an n by 3 rectangle, $n > 3$.

(i). s and t are in A or B . Match A and B . If (AB, s, t) is color compatible then it is solvable by Claim 3. There exists edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the solution of (AB, s, t) and (C, p', q') is acceptable and not forbidden. If (AB, s, t) is not color compatible, choose $\{p, p'\}$ and $\{q, q'\}$ as in Figure B.21(a) and (b), then it is shown that there is a solution to (AB, s, t) which goes through $\{p, q\}$ in Lemma B.1. (C, p', q') is acceptable and not forbidden by Claim 3.

(ii). s and t are in B or C . Match B and C . If (BC, s, t) is color compatible then it is solvable by Claim 3. There exists edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the solution of (BC, s, t) and (A, p', q') is acceptable and not forbidden. If (BC, s, t) is not color compatible, choose $\{p, p'\}$ and $\{q, q'\}$ as in Figure B.22, then it is shown that there is a solution to (BC, s, t) which goes through $\{p, q\}$ in Lemma B.1. (A, p', q') is acceptable and not forbidden by

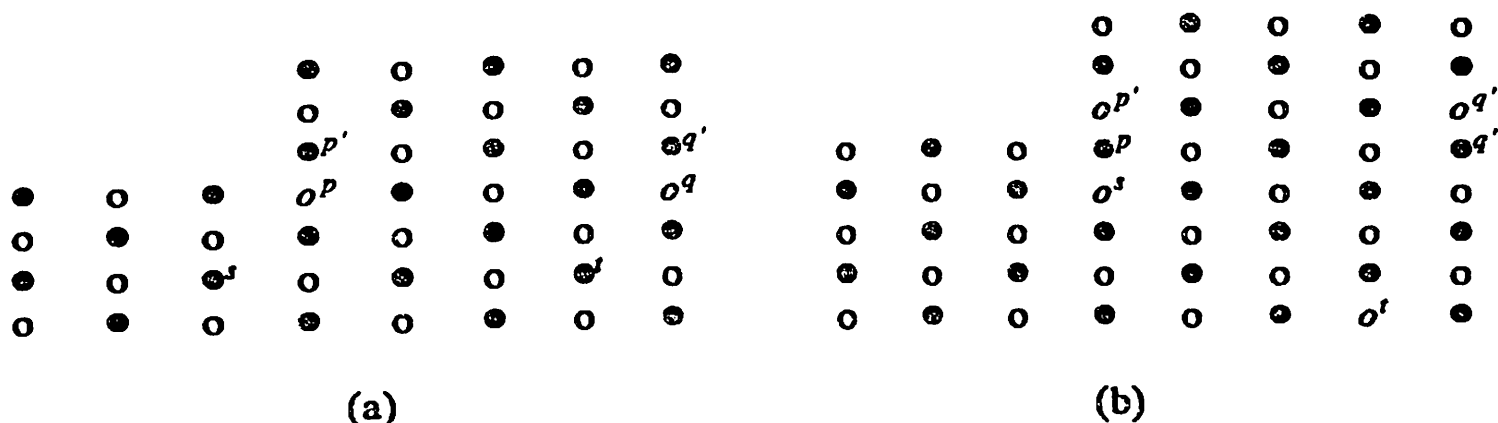


Figure B.21. A is an n by 3 rectangle, s and t in A or B .

Claim 1.

(iii). s is in A and t is in C . Match B and C . There is always a choice of $\{p, q\}$ such that (A, s, p) and (BC, q, t) are acceptable and not forbidden. Choose as in Figure B.23 unless p coincides with s . (A, s, p) and (BC, q, t) are acceptable and not forbidden by Claim 1. If p coincides with s then choose $\{p, q\}$ to be in the row above that shown. (A, s, t) is acceptable and not forbidden by Claim 1 and (BC, q, t) is acceptable and not forbidden by Claim 3. $\square$

Lemma B.5: The Hamiltonian path problem for special case problems can be solved in linear time.

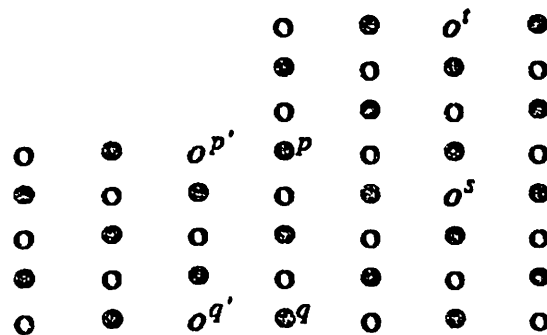


Figure B.22. A is an n by 3 rectangle, s and t are in B or C .

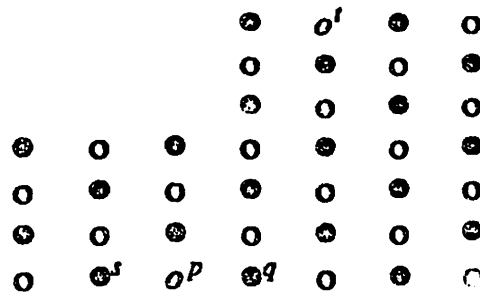


Figure B.23. A is an n by 3 rectangle, s in A , t in C .

Proof: Recall that (L^{1-3}, s, t) is a special case problem if the dimensions, (m, n) , of A are $(1, n)$, $n > 3$, and the dimensions of C are $(m, 1)$, $m > 3$. There are three cases: m and n are even, n is even and m is odd, and n and m are odd.

Case 1: n and m are even. If $n = 4$ and $m = 4$ then an exhaustive search for all possible choices of s and t has shown that there is a Hamiltonian path in all cases except in the case shown in Figure B.24 where s and t are colored differently, s is at the top left corner of B and t is not directly above or to the left of s .

If m and n are larger than four then separate L^{1-3} into two L-shaped grid graphs, $L1$ and $L2$, such that $m = 4$ and $n = 4$ in $L1$, see Figure B.25. (Note that if m or n equals four then the separation produces an L-shaped grid graph and a rectangular grid graph. The following argument holds for this case as well).

Suppose s and t are in $L1$. Since $m = 4$ and $n = 4$ in $L1$, unless it is the case shown in Figure B.24 where there is no Hamiltonian path, $(L1, s, t)$ is solvable. There must be edges $\{p, p'\}$ and $\{q, q'\}$ such that $\{p, q\}$ is on the

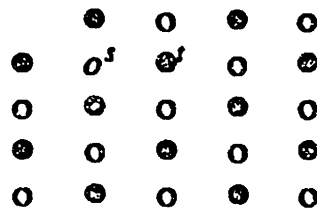


Figure B.24. A special case with no Hamiltonian path.

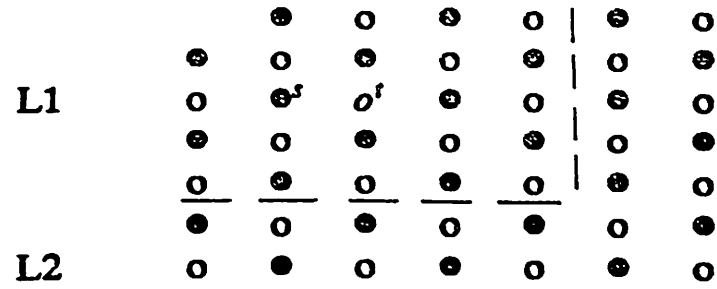
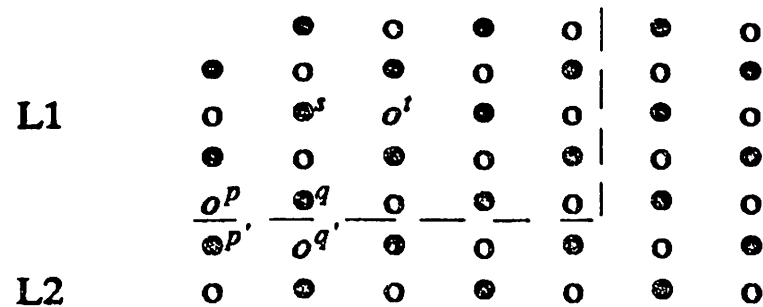


Figure B.25. Separation into L1 and L2.

solution to $(L1, s, t)$ and $(L2, p', q')$ are acceptable. $(L2, p', q')$ is solvable by Lemma B.4. Figure B.26 shows one possible selection of $\{p, p'\}$ and $\{q, q'\}$.

Suppose s is in L1 and t is in L2. There is an edge $\{p, q\}$ such that p does not coincide with s , q does not coincide with t or make $(L2, t, q)$ disconnected, and $(L1, s, p)$ and $(L2, t, q)$ are acceptable. $(L1, s, p)$ is solvable unless it is the case in Figure B.24 and Lemma B.4 shows that $(L2, t, q)$ is solvable. Figure B.27 shows one choice of $\{p, q\}$.

If s and t are in L2 and $(L2, s, t)$ is acceptable there must be edges $\{p, p'\}$ and $\{q, q'\}$ such that $(L1, p', q')$ is acceptable and $\{p, q\}$ is on the solution of $(L2, s, t)$. $(L2, s, t)$ may not be acceptable if $m = 6$ or $n = 6$. The two

Figure B.26. s and t are in L1.

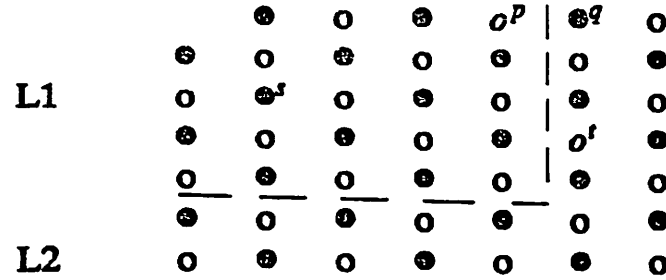
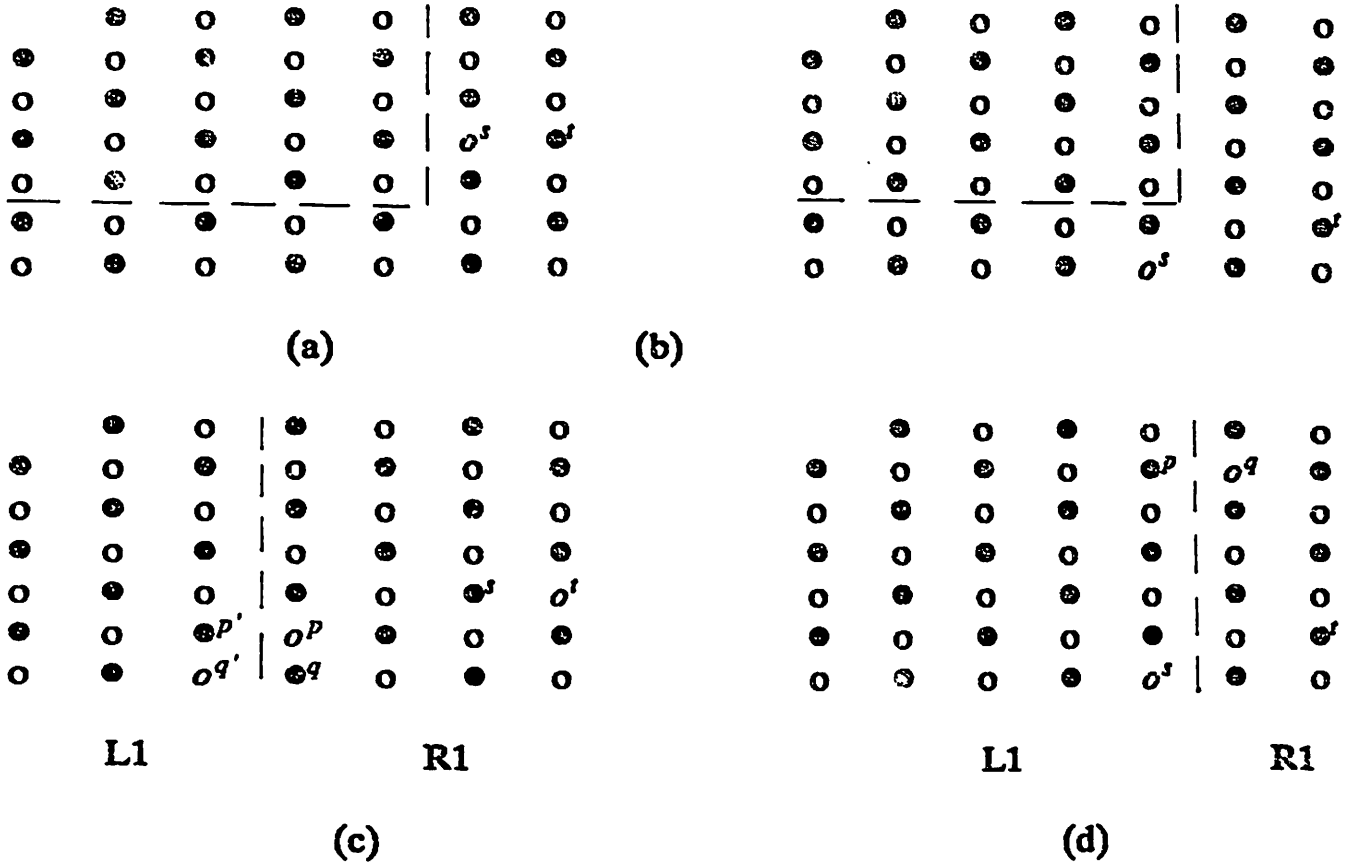


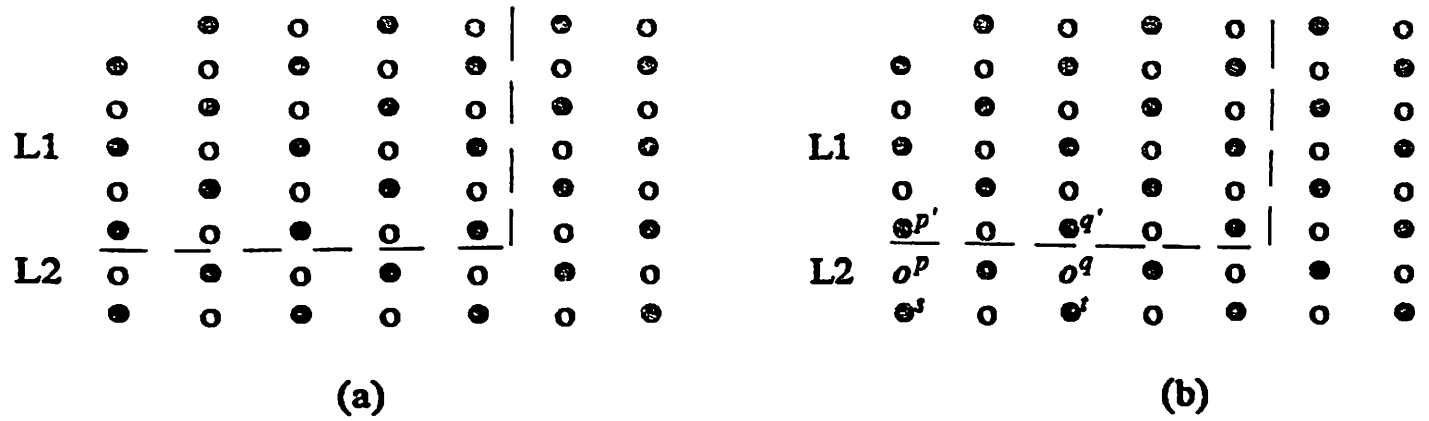
Figure B.27. s is in L2 and t is in L2.

possibilities are shown in Figure B.28(a) and (b). Consider the problem in Figure B.28(a). Separate L^{1-3} into an L-shaped grid graph, L1, and a rectangular grid graph, R1, as shown in Figure B.28(c). $(R1, s, t)$ is a solvable rectangular problem. There must be edges $\{p, p'\}$ and $\{q, q'\}$ such that $(L1, p', q')$ is solvable and $\{p, q\}$ is on the solution of $(R1, s, t)$. Consider the problem in Figure B.28(b). Separate L^{1-3} into an L-shaped grid graph, L1, and a rectangular grid graph, R1, as shown in Figure B.28(d). Choose p and q as shown. Solve $(L1, s, p)$ as above. $(R1, q, t)$ is an acceptable and not forbidden rectangular problem.

Case 2: n is even and m is odd. If $n = 4$ and $m = 5$ then an exhaustive search for all possible choices of s and t has shown that there is a Hamiltonian path in all cases. Note that s and t are the same color so s cannot be in the corner. If m and n are larger than four then separate L^{1-3} into two L-shaped graphs, L1 and L2 as shown in Figure B.29(a). Note in this case that L1 has an odd number of vertices and L2 has an even number. If s and t are in L1 then solve Case 1. If s is in L1 and t is in L2 then solve as Case 1.

Figure B.28. s and t in $L2$.

Suppose s and t are in $L2$ as shown in Figure B.29(b). In this case, since s and t are the same color and $L2$ has an even number of vertices, $(L2, s, t)$ is not color compatible. Find edges $\{p, q\}$ and $\{p', q'\}$ such that $\{p, q\}$ are in $L2$ and $\{p', q'\}$ are in $L1$ as shown in Figure B.29(b). Note that A contains an even number of rows. If A in $L2$ is four by five or larger then by Lemma B.4 and Lemma B.1 there is a Hamiltonian path from s to t which goes through edge $\{p, q\}$. If A in $L2$ is two by five then the same result can be shown by exhaustive search. Thus, $(L2, s, t)$ has a solution which goes through $\{p, q\}$. $(L1, p', q')$ is solvable by exhaustive search.

Figure B.29. n is even, m is odd.Case 3: Both n and m are odd. Solve as Case 2. $\square$